Arkhangel'skiĭ alpha properties of $C_p(X)$ and covering properties of X

Lev Bukovský

Topological Symposium, Prague, 2016

July 11, 2016

A topological space X possesses the **property** (α_i), $i = 1, 2$, see [Arh], if for any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x, there exists a sequence $\{y_m\}_{m=0}^{\infty}$ such that $\lim_{m\to\infty} y_m = x$ and

(α_1) $\{x_{n,m} : m \in \omega\} \subseteq^* \{y_m : m \in \omega\}$ for each n,

(α_2) $\{x_{n,m} : m \in \omega\} \cap \{y_m : m \in \omega\}$ is infinite for each n.

It is know that for $\mathrm{C}_p(X)$ the properties (α_2), (α_3) and (α_4) are equivalent, see [Sc3].

A topological space X possesses the **property** (α_i), $i = 1, 2$, see [Arh], if for any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x, there exists a sequence $\{y_m\}_{m=0}^{\infty}$ such that $\lim_{m\to\infty} y_m = x$ and

(α_1) $\{x_{n,m} : m \in \omega\} \subseteq^* \{y_m : m \in \omega\}$ for each n,

(α_2) $\{x_{n,m} : m \in \omega\} \cap \{y_m : m \in \omega\}$ is infinite for each n.

It is know that for $\mathrm{C}_p(X)$ the properties (α_2), (α_3) and (α_4) are equivalent, see [Sc3].

The sequence selection property **SSP**, see [Sc2], says:

for any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x, there exists a sequence $\{m_n\}_{n=0}^{\infty}$ such that $\lim_{n\to\infty} x_{n,m_n} = x$.

A sequence $\{f_n\}_{n=0}^{\infty}$ of real functions defined on a set X **quasi-normally converges** to a function f, if there exists a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of non-negative reals converging to 0 such that

$$(\forall x \in X)(\exists n_0)(\forall n \geq n_0)\,|f_n(x) - f(x)| \leq \varepsilon_n.$$

A topological space X is a **QN-space** if every sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real functions defined on X converging pointwise to 0 also quasi-normally converges to 0, [BRR].

A sequence $\{f_n\}_{n=0}^{\infty}$ of real functions defined on a set X **quasi-normally converges** to a function f, if there exists a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of non-negative reals converging to 0 such that

$$(\forall x \in X)(\exists n_0)(\forall n \geq n_0)\, |f_n(x) - f(x)| \leq \varepsilon_n.$$

A topological space X is a **QN-space** if every sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real functions defined on X converging pointwise to 0 also quasi-normally converges to 0, [BRR].

A topologocal space X is a σ-space if $\mathsf{F}_\sigma(X) = \mathrm{G}_\delta(X)$.

Theorem 1 (Recław [R])

Any perfectly normal QN*-space is a* σ*-space.*

Corollary 2 (Bukovský – Recław – Repický [BRR])

Every subset (with the subset topology) of a QN*-space is a* QN*-space.*

A topological space X is a **wQN-space** if every sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real functions defined on X converging pointwise to 0 has a subsequence $\{f_{n_k}\}_{k=0}^{\infty}$ quasi-normally converging to 0, [BRR].

A topological space X is a **wQN-space** if every sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real functions defined on X converging pointwise to 0 has a subsequence $\{f_{n_k}\}_{k=0}^{\infty}$ quasi-normally converging to 0, [BRR].

A topological space X is **perfectly meager** if every perfect subset of X is meager.

Theorem 3 (Bukovský – Recław – Repický [BRR])

Any wQN-*space is perfectly meager.*

A topological space X is a **wQN-space** if every sequence $\{f_n\}_{n=0}^\infty$ of continuous real functions defined on X converging pointwise to 0 has a subsequence $\{f_{n_k}\}_{k=0}^\infty$ quasi-normally converging to 0, [BRR].

A topological space X is **perfectly meager** if every perfect subset of X is meager.

Theorem 3 (Bukovský – Recław – Repický [BRR])

Any wQN*-space is perfectly meager.*

If $\square$ is one of the notions (α_i), $i = 1, 2$, SSP, wQN, QN, then the notion $\square_*$ or $\square^*$ is obtained by replacing continuous functions by lower or upper semicontinuous ones, respectively. See [B].

If $\square$ is one of the notions (α_i), $i = 1, 2$, SSP, then the notion $\mathrm{QN}-\square$ is obtained by replacing all cases of pointwise convergence by quasi-normal convergence. Compare [BS].

Theorem 4 (Scheepers [Sc2], Bukovský – Haleš [BH], Bukovský – Šupina [BS], Sakai [Sa2])

For a perfectly normal topological space X the following are equivalent:

$1)$ $\mathrm{C}_p(X)$ *possesses the* (α_1) *property,*

$2)$ $\mathrm{C}_p(X)$ *possesses the QN-SSP,*

$3)$ $\mathrm{C}_p(X)$ *possesses QN-*(α_2) *property,*

$4)$ $\mathrm{C}_p(X)$ *possesses* $(\alpha_2)_*$ *property,*

$5)$ $\mathrm{C}_p(X)$ *possesses the SSP*$_*$*,*

$6)$ X *is a* QN*-space,*

Theorem 4 (Scheepers [Sc2], Bukovský – Haleš [BH], Bukovský – Šupina [BS], Sakai [Sa2])

For a perfectly normal topological space X the following are equivalent:
1) $\mathrm{C}_p(X)$ *possesses the* (α_1) *property,*
2) $\mathrm{C}_p(X)$ *possesses the QN-SSP,*
3) $\mathrm{C}_p(X)$ *possesses QN-*(α_2) *property,*
4) $\mathrm{C}_p(X)$ *possesses* $(\alpha_2)_*$ *property,*
5) $\mathrm{C}_p(X)$ *possesses the SSP*$_*$*,*
6) X *is a* QN*-space,*

Corollary 5

Let X be a perfectly normal topoloical space such that $\mathrm{C}_p(X)$ *possesses the* (α_1) *property. Then*
1) X *is a perfectly meager* σ*-space,*
2) *for every subset* $Y \subseteq X$ *endowed with the subet topology the topological space* $\mathrm{C}_p(Y)$ *possesses the* (α_1) *property.*

A countable family $\{U_n : n \in \omega\}$ of subset of a set X is a γ-**cover** if $U_n \neq X$ for each n and the set $\{n \in \omega : x \notin U_n\}$ is finite for each $x \in X$.

Theorem 6 (Bukovský – Haleš [BH], Sakai [Sa1])

A perfectly normal topological space X is a QN-space if and only if the family of open γ-covers of X possesses the covering (α_1) property, i.e., for every sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of open γ-covers there exist finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_n(\mathcal{U}_n \setminus \mathcal{V}_n)$ is a γ-cover.

A countable family $\{U_n : n \in \omega\}$ of subset of a set X is a γ-**cover** if $U_n \neq X$ for each n and the set $\{n \in \omega : x \notin U_n\}$ is finite for each $x \in X$.

Theorem 6 (Bukovský – Haleš [BH], Sakai [Sa1])

A perfectly normal topological space X is a QN-space if and only if the family of open γ-covers of X possesses the covering (α_1) property, i.e., for every sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of open γ-covers there exist finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_n (\mathcal{U}_n \setminus \mathcal{V}_n)$ is a γ-cover.

A set $A \subseteq {}^{\omega}\omega$ is **eventually bounded** if A is bounded in the preorder

$$f \leq^* g \equiv (\exists n_0)(\forall n \geq n_0)\, f(n) \leq g(n).$$

Theorem 7 (Tsaban – Zdomskyy [TZ])

A perfectly normal topological space X is a QN-space if and only if every Borel measurable image of X into the Baire space ${}^{\omega}\omega$ is eventually bounded.

Theorem 8 (Fremlin [Fr], Scheepers [Sc4])

For a perfectly normal topological space X the following are equivalent:
1) $C_p(X)$ *possesses the* (α_2) *property,*
2) $C_p(X)$ *possesses the SSP,*
3) X *is a* wQN*-space.*

Theorem 8 (Fremlin [Fr], Scheepers [Sc4])

For a perfectly normal topological space X the following are equivalent:
$1)$ $C_p(X)$ *possesses the* (α_2) *property,*
$2)$ $C_p(X)$ *possesses the SSP,*
$3)$ X *is a* wQN*-space.*

A γ-cover $\mathcal{U}$ is **shrinkable** if there exists a closed γ-cover that is a refinement of $\mathcal{U}$. Γ^{sh} is the family of all open shrinkable γ-covers.

Theorem 9 (Bukovský – Haleš [BH])

If X is perfectly normal, then the following are equivalent:
$1)$ X *is a* wQN*-space,*
$2)$ X *is an* $\mathrm{S}_1(\Gamma^{\mathrm{sh}}, \Gamma)$*-space.*

Corollary 10

For a perfectly normal topological space X, $\mathrm{C}_p(X)$ *possesses the* (α_2) *property if and only if X is an* $\mathrm{S}_1(\Gamma^{\mathrm{sh}}, \Gamma)$*-space.*

Theorem 11 (Bukovský [B], Sakai [Sa2])

For a perfectly normal topological space X the following are equivalent:
$1)$ X *possesses the SSP*,*
$2)$ X *is a wQN*-space.*

Theorem 11 (Bukovský [B], Sakai [Sa2])

For a perfectly normal topological space X the following are equivalent:
$1)$ X *possesses the SSP***,*
$2)$ X *is a wQN***-space.*

Theorem 12 (Bukovsky [B])

For a perfectly normal topological space X the following are equivalent:
$1)$ X *is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space,*
$2)$ X *possesses the SSP***,*

Corollary 13

For a perfectly normal topological space X the following are equivalent:
$1)$ X *is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space,*
$2)$ X *is a wQN***-space.*

We know [BH], [Sa1] that

$$\mathrm{QN} \to \mathrm{S}_1(\Gamma, \Gamma) \to \mathrm{wQN}.$$

We know [BH], [Sa1] that

$$\mathrm{QN} \to \mathrm{S}_1(\Gamma, \Gamma) \to \mathrm{wQN}.$$

By A. Dow [D] in Laver model for Borel conjecture

$$(\alpha_2) \to (\alpha_1).$$

Thus, by Theorems 4 and 8 in Laver model

$$\mathrm{wQN} \equiv \mathrm{S}_1(\Gamma, \Gamma) \equiv \mathrm{QN}.$$

We know [BH], [Sa1] that

$$\mathrm{QN} \to \mathrm{S}_1(\Gamma, \Gamma) \to \mathrm{wQN}.$$

By A. Dow [D] in Laver model for Borel conjecture

$$(\alpha_2) \to (\alpha_1).$$

Thus, by Theorems 4 and 8 in Laver model

$$\mathrm{wQN} \equiv \mathrm{S}_1(\Gamma, \Gamma) \equiv \mathrm{QN}.$$

By W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki [JMSS]

$$if\ \mathfrak{t} = \mathfrak{b},\ then\ \mathrm{S}_1(\Gamma, \Gamma) \nrightarrow \mathrm{QN}.$$

We know [BH], [Sa1] that

$$\mathrm{QN} \to \mathrm{S}_1(\Gamma,\Gamma) \to \mathrm{wQN}.$$

By A. Dow [D] in Laver model for Borel conjecture

$$(\alpha_2) \to (\alpha_1).$$

Thus, by Theorems 4 and 8 in Laver model

$$\mathrm{wQN} \equiv \mathrm{S}_1(\Gamma,\Gamma) \equiv \mathrm{QN}.$$

By W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki [JMSS]

$$if\ \mathfrak{t} = \mathfrak{b},\ then\ \mathrm{S}_1(\Gamma,\Gamma) \not\to \mathrm{QN}.$$

Recław [R] proved that

$$there\ exists\ an\ uncountable\ \mathrm{S}_1(\Gamma,\Gamma){-}space.$$

By Miller [M]

$$It\ is\ consistent\ with\ \mathbf{ZFC}\ that\ every\ \sigma{-}set\ is\ countable.$$

Hence again, $\mathrm{S}_1(\Gamma,\Gamma) \not\to \mathrm{QN}$ is consistent with **ZFC**.

Marion Scheepers [Sc4] raised the following

Conjecture 14

For a perfectly normal topological space X

$$\mathrm{S}_1(\Gamma,\Gamma) \equiv \mathrm{wQN}, \ \textit{or}$$

$$X \ \textit{is} \ \mathrm{S}_1(\Gamma,\Gamma)\textit{-space} \equiv \mathrm{C}_p(X) \ \textit{possesses} \ (\alpha_2) \ \textit{property. or}$$

$$\mathrm{S}_1(\Gamma^{\mathrm{sh}},\Gamma) \equiv \mathrm{S}_1(\Gamma,\Gamma).$$

By presented results the conjecture is consistent with **ZFC**.

Marion Scheepers [Sc4] raised the following

Conjecture 14

For a perfectly normal topological space X

$$\mathrm{S}_1(\Gamma,\Gamma) \equiv \mathrm{wQN},\ \textit{or}$$

$$X\ \textit{is}\ \mathrm{S}_1(\Gamma,\Gamma)\textit{–space} \equiv \mathrm{C}_p(X)\ \textit{possesses}\ (\alpha_2)\ \textit{property. or}$$

$$\mathrm{S}_1(\Gamma^{\mathrm{sh}},\Gamma) \equiv \mathrm{S}_1(\Gamma,\Gamma).$$

By presented results the conjecture is consistent with **ZFC**.

However, the following is still open:

Problem 15

Is it consistent with **ZFC** *that* $\mathrm{wQN} \not\to \mathrm{S}_1(\Gamma,\Gamma)$*?*

- [Arh] Arkhangel'skiĭ A.V., *Спектр частот топологического пространства и классификация пространств*, ДАН СССР, **206**:2 (1972), 265–268. English translation: *The frequency spectrum of a topological space and the classification of spaces*, Soviet Math. Dokl. **13** (1972), 1185–1189.
- [B] Bukovský L., *On* wQN_* *and* wQN^* *spaces*, Topology Appl. **156** (2008), 24–27.
- [BH] Bukovský L. and Haleš J., *QN-spaces, wQN-spaces and covering properties*, Topology Appl. **154** (2007), 848–858.
- [BRR] Bukovský L., Recław I. and Repický M., *Spaces not distinguishing pointwise and quasi-normal convergence of real functions*, Topology Appl. **41** (1991), 25–40.
- [BS] Bukovský L. and Šupina J., *Modifications of Sequence Selection Principles*, Topology Appl. **160** (2013), 2356–2370.

[D] Dow A., *Two classes of Fréchet-Urysohn spaces*, Proc. Amer. Math. Soc. **131** (1990), 241–247.

[Fr] Fremlin D.H., SSP *and* wQN, preprint 2002.

[JMSS] Just W., Miller A.W., Scheepers M. and Szeptycki P.J., *Combinatorics of open covers (II)*, Topology Appl. **73** (1996), 241–266.

[M] Miller A.W., *Special subsets of the real line*, in: Handbook of Set Theoretic Topology (Kunen K. and Vaughan J.E., eds.), North-Holland, Amsterdam, 1984, 201 – 235.

[R] Recław I., *Metric spaces not distinguishing pointwise and quasinormal convergence of real functions*, Bull. Acad. Sci. Polonais **45** (1997), 287 – 289.

[Sa1] Sakai M., *The sequence selection properties of* $\mathrm{C}_p(X)$, Topology Appl. **154** (2007), 552–560.

[Sa2] Sakai M., *Selection principles and upper semicontinuous functions,*, Colloquium Mathematicum **117** (2009), 251–256.

[Sc1] Scheepers M., *Combinatorics of open covers I: Ramsey theory,* Topology Appl. **69** (1996), 31–62.

[Sc2] Scheepers M., *A sequential property of $C_p(X)$ and a covering property of Hurewicz,* Proc. Amer. Math. Soc. **125** (1997), 2789–2795.

[Sc3] Scheepers M., $\mathrm{C}_p(X)$ *and Arhangel'skiĭ's α_i-spaces,* Topology Appl. **89** (1998), 265–275.

[Sc4] Scheepers M., *Sequential convergences in $C_p(X)$ and a covering property,* East-West J. Math. **1** (1999), 207–214.

[TZ] Tsaban B. and Zdomskyy L., *Hereditary Hurewicz spaces and Arkhangel'skiĭ sheaf amalgamations,* J. Eur. Math. Soc. **14** (2012), 353–372.

Thank You
for your attention

Institute of Mathematics, Faculty of Sciences,
University of P.J. Šafárik
Jesenná 5, 040 01 Košice, Slovakia
e-mail: lev.bukovsky@upjs.sk