Hyperstructures in topological categories

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July 27, 2016

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- Hausdorff metric on $K(X)$ for a metric space (X, d):
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- Bourbaki uniformity on $\mathfrak{X} \subseteq \mathfrak{P}(X)$ for an uniform space $(X, \mathcal{U})$:
  $\mathcal{U}_{\mathcal{B}} := \left\{ \mathcal{S} \subseteq K(X) \times K(X) \,\middle|\, \exists R \in \mathcal{U} : \widehat{R} \subseteq \mathcal{S} \right\}$, with
  $\widehat{R} := \{(A, B) \in \mathfrak{X} \times \mathfrak{X} \mid R(A) \supseteq B \ \wedge \ R(B) \supseteq A\}$.

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- Vietoris topology on $\mathfrak{X} \subseteq \mathfrak{P}(X)$ for a topological space (X, τ):
  generated from the base $\{\langle V_1, ..., V_n \rangle_{\mathfrak{X}} \mid n \in \mathbb{N}, V_1, ..., V_n \in \tau\}$ with
  $\langle V_1, ..., V_n \rangle_{\mathfrak{X}} := \{M \in \mathfrak{X} \mid M \subseteq \bigcup_{i=1}^n V_i \ \wedge \ \forall i : M \cap V_i \neq \varnothing\}$

$$
\begin{array}{ccccc}
(X, d) & \xrightarrow{\text{unif.}} & (X, \mathcal{U}_d) & \xrightarrow{\text{topol.}} & (X, \tau_{\mathcal{U}}) \\
\Big\downarrow \text{Hausdorff} & & \Big\downarrow \text{Bourbaki} & & \Big\downarrow \text{Vietoris} \\
(K(X), d_{\mathcal{H}}) & \xrightarrow{\text{unif.}} & (K(X), \mathcal{U}_{d_{\mathcal{H}}}) & \xrightarrow{\text{topol.}} & (K(X), \tau_V)
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Problem: How to define such appropriate Hyperstructures for other kinds of spaces?

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One Idea: look for „natural“ maps between a set X and subsets $\mathfrak{X}$ of its powerset - choice functions.

Choice Functions

If X is a set and $\mathfrak{P}_0(X)$ the set of all nonempty subsets of X, let

$$\mathcal{A}(X) := \{f \in X^{\mathfrak{P}_0(X)} \mid \quad \forall A \in \mathfrak{P}_0(X) : f(A) \in A\}$$

the family of all *choice functions* on $\mathfrak{P}_0(X)$.

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One can observe, for instance:

Proposition

Let (X, τ) be a topological space, consider the lower Vietoris topology on a subset $\mathfrak{X} \subseteq \mathfrak{P}_0(X)$ and let $\widehat{\varphi}$ be an ultrafilter on $\mathfrak{X}$. Let
$P := \{p \in X \mid \exists f \in \mathcal{A}(X) : f(\widehat{\varphi}) \xrightarrow{\tau} p\}$.

1. *If there is an $A \in \mathfrak{X}$ with $A \subseteq \overline{P}$, then $\widehat{\varphi}$ converges in the lower Vietoris topology to A.*
2. *If (X, τ) is locally compact and $\widehat{\varphi}$ converges in the lower Vietoris topology to a set A, then $A \subseteq \overline{P}$ holds.*

$\mathbb{F}(M)$ = the set of all filters on a set M,
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Proposition

Let (X, τ) be a nested neighbourhood space, let $\widehat{\varphi}$ be an ultrafilter on $\mathfrak{P}_0(X)$ which converges in the lower Vietoris topology to $A \in \mathfrak{P}(X)$ and let $P := \{p \in X \mid \exists \mathcal{F} \in \mathbb{F}(\mathcal{A}(X)) : \mathcal{F}(\widehat{\varphi}) \xrightarrow{\tau} p\}$.
Then $A \subseteq P$ holds.

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Then $A \subseteq P$ holds.

Some similar things can be done for upper Vietoris convergence and so for the Vietoris itself.

For metric spaces we get even an extra nice characterization:

Theorem

Let (X, d) be a metric space, $K(X)$ the family of nonempty compact subsets of X and $d_{\mathcal{H}}$ the corresponding Hausdorff metric on $K(X)$. If $\underline{\varphi} \in \mathbb{F}(K(X))$, then the following are equivalent:

1. $\underline{\varphi} \xrightarrow{d_{\mathcal{H}}} A \in \mathfrak{X}$,
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   1. $\forall f \in \mathcal{A}(X), \underline{\psi} \in \mathbb{U}\left(\underline{\varphi}\right) : \exists a \in A : f(\underline{\psi}) \xrightarrow{d} a$ *and*
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Nevertheless definitions by choice functions need precise analyse of the concrete structure (topology, uniformity, metric ...).
Moreover, **it can lead quickly to some quite hard set theoretical difficulties:**

For a filter φ on a set X and a function $f : X \to Y$ we mean by the *image of φ under f* the filter $f(\varphi) := \{B \subseteq Y | \, \exists P \in \varphi : f[P] \subseteq B\}$.

We say, a filter Φ has *Property (A) w.r.t. X* iff Φ is a filter on $\mathfrak{P}_0(X)$ and fullfills

$$\forall f \in \mathcal{A}(X) : \exists x_f \in X : f(\Phi) = \overset{\bullet}{x_f} \tag{A}$$

(Here $\overset{\bullet}{x_f} := \{A \subseteq X \mid x_f \in A\}$ is the *singleton filter* generated by x_f.)

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Question: If Φ has property (A) w.r.t. X, must Φ itself be a singleton filter on $\mathfrak{P}_0(X)$?

Proposition

If a filter Φ *has property (A) w.r.t. a set* X*, then it is an ultrafilter on* $\mathfrak{P}_0(X)$*.*

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Lemma

If Φ has property (A) w.r.t. a set X, then Φ is **countably complete***.*

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Lemma

If Φ has property (A) w.r.t. a set X, then Φ is **countably complete**.

Corollary

If Φ has property (A) w.r.t. a **countable** *set X, then Φ is a singleton filter on $\mathfrak{P}_0(X)$.*

Now the bad news:

1. Countably complete **free** ultrafilter exist, iff ω-measurable cardinals exist.
2. ω-measurable cardinals exist, iff measurable cardinals exist.

The problem:

4. In ZFC+„there exists an inaccessible cardinal" the consistency of ZFC can be proved.

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$\Longrightarrow$ no hope to prove the existence of free ultrafilters with property (A) within ZFC.

Interesting Questions:

1. Can we prove in ZFC anyway, that free ultrafilters with property (A) do **not** exist?
2. If Φ is a filter on $\mathfrak{P}_0(X)$ such that for every $f \in \mathcal{A}(X)$ the image $f(\Phi)$ is an ultrafilter on X. Must Φ itself be an ultrafilter on $\mathfrak{P}_0(X)$?

Now we take category theory into account: hoping to find a categorical *description* of the Vietoris topology.

Topological Categories

A concrete category $\mathcal{C}$ over **Set** is called *topological*, iff

1. For all $X \in |\mathbf{Set}|$ and all families $(f_i, (X_i, \xi_i))_{i \in I}$, indexed by a class I, of $\mathcal{C}$-objects (X_i, ξ_i) and functions $f_i : X \to X_i$ there exists a unique initial $\mathcal{C}$-Object (X, ξ) on the set X, i.e. an object (X, ξ) s.t. for all objects $(Y, \eta) \in |\mathcal{C}|$ and maps $g : Y \to X$ holds

$$g \in \mathbf{Mor}((Y,\eta),(X,\xi))_{\mathcal{C}} \quad \Leftrightarrow \quad \forall i \in I : f_i \circ g \in \mathbf{Mor}((Y,\eta),(X_i,\xi_i))_{\mathcal{C}}$$

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   $$(Y, \eta) \xrightarrow{g} (X, \xi) \xrightarrow{f_i} (X_i, \xi_i)$$

   That is: arbitrary initial structures exist. Note that this is equivalent to the existence of arbitrary *final structures*:

   $$(Z, \zeta) \xleftarrow{g} (X, \xi) \xleftarrow{f_i} (X_i, \xi_i)$$

2. (Fibre-smallness) For all $X \in |\mathbf{Set}|$, the class of $\mathcal{C}$-objects on X is a set.
3. On sets with at most one element exists exactly one $\mathcal{C}$-structure.

Improvement: cartesian closedness

A category $\mathcal{C}$ is called **cartesian closed**, iff

4.
   1. For every pair (A, B) of $\mathcal{C}$-objects exists a product $A \times B$ in $\mathcal{C}$ and
   2. For every pair (A, B) of $\mathcal{C}$-objects exists a $\mathcal{C}$-object B^A and a $\mathcal{C}$-morphism $e : A \times B^A \to B$, s.t. for every $\mathcal{C}$-Object C and every $\mathcal{C}$-morphism $f : A \times C \to B$ there exists a unique $\mathcal{C}$-morphism $\overline{f} : C \to B^A$ with $f = e \circ (\mathbb{1}_A \times \overline{f})$.

that is: $\mathcal{C}$ has „natural function spaces“.

A topological category $\mathcal{C}$ is said to be **extensional**, iff for every $\mathbf{Y} \in |\mathcal{C}|$ with underlying set Y, there exists a $\mathcal{C}$-object $\mathbf{Y}^*$ with underlying set $Y^* := Y \cup \{\infty_Y\}$, $\infty_Y \notin Y$, s.t. for every $\mathbf{X} \in \mathcal{C}$ with underlying set X, every $Z \subseteq X$ and every $f : Z \to Y$, where f is a $\mathcal{C}$-morphism w.r.t. the subobject $\mathbf{Z}$ of $\mathbf{X}$ on Z, the map $f^* : X \to Y^*$, defined by

$$f^*(x) := \begin{cases} f(x) & ; \quad x \in Z \\ \infty_Y & ; \quad x \notin Z \end{cases}$$

is a $\mathcal{C}$-morphism.
that means: $\mathcal{C}$ has „one-point-extensions“

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A topological category $\mathcal{C}$ is called a **topological universe**, iff it is cartesian closed and extensional.

Top and **Unif** are topological categories, but not cartesian closed and not extensional.

Hyperspaces and function spaces

There are well known connections between hyperspaces and function spaces, for instance:

- graph topologies (Naimpally, Poppe, ...)
- function spaces on characteristic functions of subsets (Flachsmeyer, Poppe, ...)

There is another one, that we want to propose here for investigation.

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There is another one, that we want to propose here for investigation.

We start with a function space structure:
Let X be a set and (Y, σ) a topological space. For $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ we call the topology on Y^X generated by the subbase of all sets

$$(A, O) := \{f \in Y^X \mid \quad f(A) \subseteq O\}$$

with $A \in \mathfrak{A}$ and $O \in \sigma$ the $\mathfrak{A}$-open topology on Y^X (or on $C(X, Y)$, if X has a topology, too, or other subsets of Y^X).

We define a mapping μ_X from Y^X to $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ by

$$\forall M \in \mathfrak{A} : \quad \mu_X(f)(M) := f[M].$$

Lemma

Let $(X,\tau),(Y,\sigma)$ *be topological spaces, let* $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ *contain the singletons and* $\mathcal{H} \subseteq Y^X$. *Then the map*

$$\mu_X : \mathcal{H} \to \mu_X(\mathcal{H}) := \{\mu_X(f) |\ f \in \mathcal{H}\} \subseteq \mathfrak{P}_0(Y)^{\mathfrak{A}}$$

is open, continuous and bijective, where $\mathcal{H}$ *is equipped with the* $\mathfrak{A}$*-open topology and* $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ *with the pointwise from the Vietoris topology on* $\mathfrak{P}_0(Y)$.

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Note:

1. For $\mathfrak{A} = K(X)$ (the family of nonempty compact subsets of X) we get the compact-open topology on $\mathcal{H} := C(X,Y)$.

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2. For locally compact (X,τ) the compact-open topology induces the convergence structure of continuous convergence on $C(X,Y)$.

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Note:

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2. For locally compact (X, τ) the compact-open topology induces the convergence structure of continuous convergence on $C(X, Y)$.
3. The continuous convergence is the „natural“ function space structure in the *topological universe* **PsTop**.

We have:

$$\begin{array}{ccc} C(X,Y) \xrightarrow{\mu_X} K(Y)^{K(X)} & \cong & \prod_{A \in K(X)} K(Y)_A \\ & & \big\downarrow \pi_A \\ & & K(Y) \end{array}$$

where π_A are the canonical projections, $C(X, Y)$ is endowed with compact-open topology (which is the natural function space structure, whenever X is locally compact) and $K(Y)$ is endowed with Vietoris topology.

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Then the functions $\pi_A \circ \mu_X$, for all domain spaces X and all compact subsets A of Y are *all continuous*.

Question: Is the Vietoris topology on $K(Y)$ the final topology w.r.t this family of functions?

Lemma

Let (X,τ), (Y,σ) be topological spaces and let σ_V be the Vietoris topology on $K(Y)$. Then for every $\mathfrak{O} \in \sigma_V$ and every $A \in K(X)$ the set $(\pi_A \circ \mu_X)^{-1}(\mathfrak{O}) \subseteq C(X, Y)$ is open w.r.t. the compact-open topology.

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Corollary

Let (Y,σ) be a topological space. For every topological space let $C(X,Y)$ be equipped with compact-open topology.
Then the Vietoris topology σ_V on $K(Y)$ is contained in the final topology w.r.t. all $\pi_A \circ \mu_{(X,\tau)}$, $(X,\tau) \in \mathcal{B}$, $A \in K(X,\tau)$, for every class $\mathcal{B}$ of topological spaces.

Theorem

Let (Y, σ) be a T_3-space and let $(K(Y), \sigma_V)$ be its Vietoris Hyperspace of compact subsets. Let furthermore δ be the discrete topology on $Y \times Y$ and denote by (Z, ζ) the Stone-Čech-compactification of $(Y \times Y, \delta)$. Then σ_V is the final topology on $K(Y)$ w.r.t. $\pi_Z \circ \mu_Z : C(Z, Y) \to K(Y)$, where $C(Z, Y)$ is endowed with compact-open topology τ_{co}.

Corollary

Let (Y, σ) be a T_3-space. For every topological space let $C(X, Y)$ be equipped with compact-open topology. Let $\mathcal{B}$ be a class of topological spaces, that contains the Stone-Čech-compactification of a discrete space with cardinality at least card(Y).
Then the Vietoris topology σ_V on $K(Y)$ is the final topology w.r.t. all $\pi_A \circ \mu_{(X,\tau)}$, $(X, \tau) \in \mathcal{B}$, $A \in K(X, \tau)$.

We get also a description for the Vietoris hyperspace of the closed subsets.

Lemma

Let (Y, σ) be a Hausdorff T_4-space. Then its Vietoris hyperspace on the nonempty closed subsets $(Cl(Y), \sigma_V)$ is homeomorphic to a subspace of the Vietoris hyperspace $(K(\beta Y), \sigma^\beta)$ of compact subsets of the Stone-Čech-compactification of (Y, σ).

A topological universe containing **Unif**

For sets X we define a relation $\preceq$ between elements of $\mathfrak{P}_0(\mathfrak{P}_0(X))$:

$$\alpha_1 \preceq \alpha_2 :\Leftrightarrow \forall A_1 \in \alpha_1 : \exists A_2 \in \alpha_2 : A_1 \subseteq A_2 \ .$$

For subsets $\Sigma_1, \Sigma_2 \subseteq \mathfrak{P}_0(\mathfrak{P}_0(X))$:

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$\preceq$ is reflexive and transitive, but not symmetric, not antisymmetric and not asymmetric in general.

Definition multifilter

Let X be a set. A family $\Sigma \subseteq \mathfrak{P}_0(\mathfrak{P}_0(X))$ is called a **multifilter** on X, iff

1. $\sigma_1 \in \Sigma \wedge \sigma_1 \preceq \sigma_2 \implies \sigma_2 \in \Sigma$ and
2. $\sigma_1, \sigma_2 \in \Sigma \implies \exists \sigma_3 \in \Sigma : \sigma_3 \preceq \sigma_1$ and $\sigma_3 \preceq \sigma_2$

hold. The set of all multifilters on a set X we denote by $\widehat{\mathfrak{F}}(X)$.

Examples: Every uniformity in the covering sense (Tukey) is a multifilter.
For $x \in X$ the family $\widehat{x} := \{\sigma \subseteq \mathfrak{P}_0(X) | \ \{\{x\}\} \preceq \sigma\}$ is a multifilter.

Let $x \in X$ and $\alpha \subseteq \mathfrak{P}_0(X)$. Then the *star of α at x* is defined as

$$st(x, \alpha) := \bigcup_{A \in \alpha, x \in A} A \ ,$$

and the *weak star set* of α at x is defined as

$$\Diamond(x, \alpha) := \{ \bigcup_{i=1}^{n} A_i | \ n \in \mathbb{N}, \forall i = 1, ..., n : x \in A_i \in \alpha \} \ .$$

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For a partial cover σ of a set X let
$\sigma^{\Diamond} := \bigcup_{x \in X, \Diamond(x,\sigma) \neq \varnothing} \Diamond(x, \sigma)$,
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$\Sigma^{\Diamond} := \{\xi \in \mathfrak{P}_0(\mathfrak{P}_0(X)) | \ \exists \sigma \in \Sigma : \sigma^{\Diamond} \preceq \xi\}$,
$\Sigma^* := \{\xi \in \mathfrak{P}_0(\mathfrak{P}_0(X)) | \ \exists \sigma \in \Sigma : \sigma^* \preceq \xi\}$.

Definition multifilter-space

For a set X and a set $\mathcal{M}$ of multifilters on X we call the pair $(X, \mathcal{M})$ a **multifilter-space**, iff

1. $\forall x \in X : \widehat{x} \in \mathcal{M}$ and
2. $\Sigma_1 \in \mathcal{M} \wedge \Sigma_2 \preceq \Sigma_1 \Rightarrow \Sigma_2 \in \mathcal{M}$

hold. $\mathcal{M}$ is called the **multifilter-structure** of this space.
If $(X_1, \mathcal{M}_1)$, $(X_2, \mathcal{M}_2)$ are multifilter-spaces and $f : X_1 \to X_2$ is a map, then f is called **fine** (w.r.t. $\mathcal{M}_1, \mathcal{M}_2$), iff

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3. *weakly uniform* iff $\forall \Sigma \in \mathcal{M} : \Sigma^{\lozenge} \in \mathcal{M}$,
4. *uniform* iff $\forall \Sigma \in \mathcal{M} : \Sigma^{*} \in \mathcal{M}$.

Lemma

The multifilter-spaces as objects and the fine mappings between them as morphisms form a strong topological universe, denoted by **MFS**. The natural function-space between the multifilter-spaces $\mathbf{X} := (X, \mathcal{M})$ and $\mathbf{Y} := (Y, \mathcal{N})$ is $(\mathbf{Y}^{\mathbf{X}}, \mathcal{M}_{\mathbf{X},\mathbf{Y}})$ with $\mathcal{M}_{\mathbf{X},\mathbf{Y}} := \{\Gamma \in \widehat{\mathfrak{F}}(\mathbf{Y}^{\mathbf{X}}) | \, \forall \Sigma \in \mathcal{M} : \Gamma(\Sigma) \in \mathcal{N}\}$.

The subcategories of limited, principal, weak uniform limited, weak uniform principal, uniform limited and uniform principal multifilter-spaces are denoted by **LimMFS**, **PrMFS**, **WULimMFS**, **PrWULimMFS**, **ULimMFS** and **PrULimMFS**, respectively.

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The category **UMer** of uniform covering spaces (in the sense of Tukey) and uniformly continuous maps is concretely isomorphic to **PrULimMFS**.

$A_1, ..., A_n \subseteq X$, $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$:

$$< A_1, ..., A_n >_{\mathfrak{A}} := \{M \in \mathfrak{A} |\ M \subseteq \bigcup_{i=1}^{n} A_i \wedge \forall i = 1, ..., n : M \cap A_i \neq \varnothing\}$$

For $\alpha \subseteq \mathfrak{P}_0(X)$ we set $\alpha_{V,\mathfrak{A}} := \{< A_1, ..., A_n > \mid n \in \mathbb{N}, A_i \in \alpha\}$ and for $\Sigma \in \widehat{\mathfrak{F}}(X)$ we define $\Sigma_{V,\mathfrak{A}} := [\{\alpha_{V,\mathfrak{A}} |\ \alpha \in \Sigma\}]_{\widehat{\mathfrak{F}}(\mathfrak{A})}$.

Definition *finite hyperstructure*

Let $(X, \mathcal{M})$ be a limited multifilter-space. Then we call

$$\mathcal{M}_V := \{\underline{\Sigma} \in \widehat{\mathfrak{F}}(\mathcal{PC}(X)) |\ \exists \Psi \in \mathcal{M} : \underline{\Sigma} \preceq \Psi_{V,\mathcal{PC}(X)}\}$$

the **finite hyperstructure** on $\mathcal{PC}(X)$ w.r.t. $\mathcal{M}$.

If $(X, \mathcal{M})$ is a limited multifilter-space, then $(\mathcal{PC}(X), \mathcal{M}_V)$ is a limited multifilter-space, too.

This hyperstructure is build very Vietoris-like and works fine in some sense:

Theorem

Let $(X, \mathcal{M})$ be a limited multifilter-space. Then $(\mathcal{PC}(X), \mathcal{M}_V)$ is precompact, if and only if $(X, \mathcal{M})$ is precompact.

Lemma

If $(X, \mathcal{M})$ is a limited multifilter-space and $\mathfrak{A} \subseteq \mathcal{PC}(X)$, then $\mathfrak{A}$ is precompact w.r.t. $\mathcal{M}_V$ if and only if $\bigcup_{A \in \mathfrak{A}} A$ is precompact w.r.t. $\mathcal{M}$.

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But: it is *not* the final multifilterstructure on $\mathcal{PC}(X)$ w.r.t. all situations

$$(\mathbf{Y}^{\mathbf{X}}, \mathcal{M}_{\mathbf{X},\mathbf{Y}}) \xrightarrow{\mu_X} \mathcal{PC}(Y)^{\mathcal{PC}(X)} \xrightarrow{\pi_A} \mathcal{PC}(Y),$$

although the map μ_X is always a morphism, too.

Thank you for your attention!