An internal characterization for productively Lindelöf spaces

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[1]Supported by FAPESP

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Problem (Tamano)

Is there an internal characterization for productively Lindelöf spaces?

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- Since we are talking about open coverings of X, this can be said in terms of X.
- Actually, we can do a little better - we do not need to say that $X \times L$ is Lindelöf. We only have to check if a very specific (and simple) covering has a countable subcovering - we will see that later.

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Let $C \in \mathcal{C}_X$. A basic open neighborhood of C is of the form $[A_1, ..., A_n]$ for $A_1, ..., A_n \in \mathcal{C}_X$, where

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Note that this topology is quite natural: two open coverings are more close to each other as many open set they share.

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Theorem (easier to remember version)

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As said before, we do not need $X \times L$ being Lindelöf - just a simple consequence of it.

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Theorem (a more internal version)

A topological space X is productively Lindelöf if, and only if, for every Lindelöf collection L of open coverings of X, there is a sequence $(A_n)_{n\in\omega}$ of open sets such that $C \cap \{A_n : n \in \omega\}$ is an open covering for every $C \in L$.

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- Since $X \times L$ is Lindelöf, there is a sequence $(A_n)_{n \in \omega}$ such that $X \times L \subset \bigcup_{n \in \omega} A_n \times [A_n]$.
- Note that, for a fixed $C \in L$, for every $x \in X$, there is an $n \in \omega$ such that $(x, C) \in A_n \times [A_n]$. Thus, this is simply saying that $C \cap \{A_n : n \in \omega\}$ is a covering.

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- The sequence $(A_n)_{n\in\omega}$ induces a countable subcovering of $\mathcal{W}$.

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Thus, note that if the $\mathcal{W}$ we were working with is ω-good, we are done: at the end, when we fix the sequence $(A_n)_{n\in\omega}$, the set

$$\{A_n \times B : A_n \times B \in \mathcal{W}\}$$

is countable and would be a covering.

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- Note that if X is discrete, the result is true;

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- Work by induction on the cardinality of the elements of this base;
- Be patient.

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One way of not having this problem is to change the way that we made the L space. Instead of making it with a collection of open coverings, we could made it with indexed open coverings:

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$$[i_1, ..., i_n] = \{y \in \mathcal{Y} : i_1, ..., i_n \in y\}$$

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With this, we can say when $\mathcal{Y}$ is Lindelöf or is not and thus we can repeat the previous process. Note that in this way, we do not have the final problem: we can always take coverings $(A_\xi \times B_\xi)_{\xi < \kappa}$ indexed by some κ. Working in this way, associated with each A_ξ, there is only one B_ξ.

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With indexed families (not necessarily open coverings), we can repeat this idea and obtain internal characterizations for the productiveness of other topological properties.

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Let's see some examples.

Countably compact spaces

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A space X is productively countably compact if, and only if, for every space L of indexed coverings that is countably compact, $X \times L$ is countably compact.

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In the Lindelöf case, this was not a problem since, by a result of Duanmu, Tall and Zdomskyy, if $X \times Y$ is not Lindelöf for some Y Lindelöf, there is a regular Lindelöf space Z such that $X \times Z$ is not Lindelöf.

Menger and Rothberger spaces

Proposition

A space X is productively Rothberger (Menger) if, and only if, $X \times L$ is Rothberger (Menger) for every Rothberger (Menger) space L made of indexed coverings of X.

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The trick here is when taking an open covering $\mathcal{W}$ of $X \times Y$, we refine this covering for a collection $(A_i \times B_i)_{i \in I}$ in such a way that the collection $\{B_i : i \in I\}$ forms a base for Y (this let us pass through the "refinement problems").

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- For each $y \in Y$, let $C_y = \{A_\xi : y \in B_\xi\}$. Let $L = \{C_y : y \in Y\}$.
- Now we have to prove that L is paracompact and, after this, prove that a locally finite refinement from $A_\xi \times [A_\xi]$ induces a locally finite refinement for $(A_\xi \times B_\xi)_{\xi<\kappa}$.

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- Since Y is paracompact, there is $(U_s)_{s\in S}$ a locally finite refinement of $(V_{\overline{\xi}})_{\overline{\xi}\in I}$.

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- Since Y is paracompact, there is $(U_s)_{s\in S}$ a locally finite refinement of $(V_{\overline{\xi}})_{\overline{\xi}\in I}$.
- For each $s \in S$, let $\kappa_s = \{\xi < \kappa : B_\xi \subset U_s\}$.
- For each $s \in S$, let $A^s = \bigcup_{\xi\in\kappa_s}[A_\xi]$. The collection $\{A_s : s \in S\}$ is the locally finite refinement we were looking for.

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For each $[A_\xi^n, ..., A_\xi^n]$ as above, define $W_\xi = \bigcap_{i=1}^n B_\xi^n$.

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Then, for each s, define $\mathcal{U}_s = \bigcup_{\xi\in I_S} V_\xi \times W_\xi$.
$(\mathcal{U}_s)_{s\in S}$ is the refinement that we were looking for.

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Proposition

A space X is productively Baire if, and only if, $X \times L$ is Baire for every Baire space L made of indexed open collections of X.

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A space X is productively Baire if, and only if, $X \times L$ is Baire for every Baire space L made of indexed open collections of X.

The trick here is the translation: for a open set A, A is dense if, and only if, $\{B \subset A : B \text{ is open}\}$ is a π-base.

ccc spaces

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Proposition

A space X is productively ccc if, and only if, for any family $\mathcal{A}$ of antichains of X with $|\bigcup \mathcal{A}| > \aleph_0$, there exists an uncountable set $\mathcal{F} \subset \bigcup_{A \in \mathcal{A}} [A]^{<\omega}$ such that there is no $F, G \in \mathcal{F}$ with $F \neq G$ and $F \cup G \in A$ for some $A \in \mathcal{A}$.

Thank you very much