Many weak \( P \)-sets

Jan van Mill\(^1\)

University of Amsterdam

Thirteenth Symposium on General Topology and its Relations to Modern Analysis and Algebra
July 25, 2022

\(^1\)Joint work with Alan Dow
Many weak $P$-sets

Fourth Symposium on General Topology 
and its Relations to Modern Analysis and Algebra

was held on August 23-27, 1976 in Prague, Czech Republic. It was organized by the Mathematical Institute of the Czechoslovak Academy of Sciences with support of the International Mathematical Union and in cooperation with the Slovak Academy of Sciences, the Faculty of Mathematics and Physics of the Charles University and the Association of Czechoslovak Mathematicians and Physicists.

Organizing committee

- J. Novák (chairman)
- A. Rážek (treasurer)
- Z. Frolík
- J. Hejcman
- M. Hušek
- M. Katětov
- V. Koutník
- V. Pták
- S. Schwarz
- M. Sekanina
- V. Trnková

The Symposium was attended by 217 mathematicians from 24 countries, including 53 from Czechoslovakia. The program consisted of 30 invited talks (11 plenary, 18 semiplenary, 1 in a session for contributed papers), and 135 fifteen minute talks in three or four parallel sessions.

Prague 1976
### Document from 1976, 42 years ago

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22.8.1976

Chátrku a podpis
Many weak $P$-sets

1961

61 years ago!
Many weak $P$-sets
Many weak $P$-sets

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<td><strong>Bessaga, Pełczyński</strong> (Poland)</td>
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<td><strong>Borsuk, Engelking</strong> (Poland)</td>
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<td><strong>Dowker, Mazur</strong> (UK)</td>
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<td>de Groot (the Netherlands)</td>
<td><strong>Kuratowski</strong> (Poland)</td>
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<td><em>Isbell, Klee, Wallace</em> (USA)</td>
<td><strong>Lejeune, Hakim</strong> (France)</td>
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<td><strong>Nagata, Shirotot</strong> (Japan)</td>
<td><strong>Chogoshvili</strong> (Georgia)</td>
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The Symposium was held in an atmosphere of friendship and contributed to the establishment and strengthening of personal contacts between the scientists from different countries.

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**REPORT OF THE ORGANIZING COMMITTEE**

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- **Let us express hope that the war in Ukraine will not result in such a division again!**
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- \( \beta \mathbb{N} \) is the \textit{Stone space} \( \text{st}(\mathcal{P}(\mathbb{N})) \) of the Boolean algebra \( \mathcal{P}(\mathbb{N}) \) (Hence \( \beta \mathbb{N} \) is \textit{zero-dimensional}).
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- $\beta\mathbb{N}$ surfaces at many places in mathematics: topology, set theory, logic, analysis, algebra, etc.
- In the ‘old’ days there was a lot of interest in the individual *points* of $\beta\mathbb{N}$.
- Walter Rudin proved that $\mathbb{N}^*$ is not *homogeneous* under CH. That is, there are two points in $\mathbb{N}^*$ that have different topological behavior in $\mathbb{N}^*$. Frolík proved this in ZFC. Shelah proved that Rudin’s method does not work in ZFC alone.
A definitive result was proved by Kunen in 1978: $\mathbb{N}^*$ contains a so-called weak $P$-point. That is a point $p \in \mathbb{N}^*$ such that $p \notin \overline{A}$, where $A$ is any countable subset of $\mathbb{N}^* \setminus \{p\}$. 
A definitive result was proved by Kunen in 1978: \( \mathbb{N}^* \) contains a so-called *weak P-point*. That is a point \( p \in \mathbb{N}^* \) such that \( p \notin \overline{A} \), where \( A \) is any countable subset of \( \mathbb{N}^* \setminus \{p\} \).

If \( A \subseteq \mathbb{N}^* \) is any countably infinite set, then there exists \( q \in \overline{A} \setminus A \), hence \( q \) is not a weak P-point.
All points in $\mathbb{N}^*$ are topologically homeomorphic (deep theorem), but there are points $p$ and $q$ in $\mathbb{N}^*$ with obvious different topological behavior.
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Kunen’s brilliant proof was generalized in several directions. For example instead of \textit{weak P-points} one can create certain \textit{weak P-sets} in \( \mathbb{N}^* \) (Dow, Simon, vM).
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For the ’interesting subspaces’ $A$ and $B$ of $\mathbb{N}^*$, we can ask:

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2. If they are, are they placed in the same way in $\mathbb{N}^*$?
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Are there such subspaces, besides $\mathbb{N}^*$ itself?
Every proper nonempty clopen subspace of $\mathbb{N}^*$ is homeomorphic to $\overline{\mathbb{N}}^*$. 

Van Douwen called such copies of $\mathbb{N}^*$ in $\mathbb{N}^*$ trivial.

Around 1980 (our best guess) he asked: is there a nowhere dense copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is not trivial?

Reformulating: is there a nowhere dense copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is not placed in $\mathbb{N}^*$ in a trivial way?
Many weak $P$-sets

- Every proper nonempty clopen subspace of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$.
- Are there copies of $\mathbb{N}^*$ in $\mathbb{N}^*$ that have empty interior in $\mathbb{N}^*$?

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Reformulating: is there a nowhere dense copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is not placed in $\mathbb{N}^*$ in a trivial way?
**Theorem (Dow (2014))**

There is a nontrivial nowhere dense copy of $\mathbb{N}^*$ in $\mathbb{N}^*$. 

1. An Aronszajn tree is a tree of uncountable height with no uncountable branches and no uncountable levels.
2. Here 'nice' means that for every $F \in \mathcal{F}$, the set $\{ n \in \mathbb{N} : F \cap (\{n\} \times 2^{\omega}) = \emptyset \}$ is finite.
Many weak $P$-sets

**Theorem (Dow (2014))**

There is a nontrivial nowhere dense copy of $\mathbb{N}^*$ in $\mathbb{N}^*$.

- Dow used an Aronszajn tree in $2^{<\omega_1}$ to prove the existence of a so-called *nontrivial, maximal, nice* closed filter $\mathcal{F}$ on $\mathbb{N} \times 2^{\omega_1}$. 

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1. Here ‘nontrivial’ means that for all $x_n \in 2^{\omega_1}$, $n \in \mathbb{N}$, there exists $F \in F$ such that $\{n \in \mathbb{N} : (n, x_n) \notin F\}$ is infinite.
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  Here ‘maximal’ means that if for every $n \in \mathbb{N}$, $\{C_0^n, C_1^n\}$ is a clopen partition of $2^{\omega_1}$, there exist $F \in \mathcal{F}$ and $f \in 2^\mathbb{N}$ such that for every $n$, $F \cap (\{n\} \times 2^{\omega_1}) \subseteq \{n\} \times C^n_{f(n)}$.
Let $Y = \beta(\mathbb{N} \times 2^{\omega_1})$, the Čech-Stone compactification of $\mathbb{N} \times 2^{\omega_1}$. 
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Then, as Dow showed, \( K_F = \bigcap_{F \in \mathcal{F}} \overline{F} \) is a ‘nontrivial’ copy of \( \mathbb{N}^* \) in \( \beta Y \).
Many weak $P$-sets

Let $Y = \beta(\mathbb{N} \times 2^{\omega_1})$, the Čech-Stone compactification of $\mathbb{N} \times 2^{\omega_1}$.

Then, as Dow showed, $K_F = \bigcap_{F \in \mathcal{F}} \overline{F}$ is a ‘nontrivial’ copy of $\mathbb{N}^*$ in $\beta Y$.

We are not done since $Y$ does not embed in $\mathbb{N}^*$. 

So instead of in $2^{\omega_1}$, Dow used $E(2^{\omega_1})$, the projective cover (or absolute) of $2^{\omega_1}$. It is an extremally disconnected compact separable space of weight $c$.

Each node of the Aronszajn tree is associated to a ‘compatible’ ultrafilter of regular open sets in some $2^{\alpha}$, for $\alpha < \omega_1$.

This allowed Dow to do the same thing as above in $\beta(\mathbb{N} \times E(2^{\omega_1}))$ instead of $\beta(\mathbb{N} \times 2^{\omega_1})$.

Kunen’s machinery of constructing a weak $P$-point in $\mathbb{N}^*$ is used to embed $\beta(\mathbb{N} \times E(2^{\omega_1}))$ as a weak $P$-set in $\mathbb{N}^*$. 
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Each node of the Aronszajn tree is associated to a ‘compatible’ ultrafilter of regular open sets in some $2^\alpha$, for $\alpha < \omega_1$. 
Let $Y = \beta(N \times 2^{\omega_1})$, the Čech-Stone compactification of $N \times 2^{\omega_1}$.

Then, as Dow showed, $K_\mathcal{F} = \bigcap_{F \in \mathcal{F}} \overline{F}$ is a ‘nontrivial’ copy of $N^*$ in $\beta Y$.

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- Then, as Dow showed, $K = \cap_{F \in \mathcal{F}} \overline{F}$ is a ‘nontrivial’ copy of $N^*$ in $\beta Y$.
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- Each node of the Aronszajn tree is associated to a ‘compatible’ ultrafilter of regular open sets in some $2^\alpha$, for $\alpha < \omega_1$.
- This allowed Dow to do the same thing as above in $\beta(N \times E(2^{\omega_1}))$ instead of $\beta(N \times 2^{\omega_1})$.
- Kunen’s machinery of constructing a weak $P$-point in $N^*$ is used to embed $\beta(N \times E(2^{\omega_1}))$ as a weak $P$-set in $N^*$.
This gives a nontrivial copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is contained in a separable closed subspace of $\mathbb{N}^*$. 
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- This gives a nontrivial copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is contained in a separable closed subspace of $\mathbb{N}^*$.
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Theorem (Dow and vM)

There is copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is a nowhere dense weak $P$-set.

Instead of $E(2^{\omega_1})$ we use the Stone space of the measure algebra $M^{\omega_1}$ on $2^{\omega_1}$. 
• This gives a nontrivial copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is contained in a separable closed subspace of $\mathbb{N}^*$.
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**Theorem (Dow and vM)**

*There is copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is a nowhere dense weak $P$-set.*

Instead of $E(2^{\omega_1})$ we use the Stone space of the measure algebra $\mathcal{M}_{\omega_1}$ on $2^{\omega_1}$. 
Many weak $P$-sets

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We let every node in the Aronszajn tree that we used before correspond to a ‘compatible’ remote point in the Stone space of a certain subalgebra of $\mathcal{M}_{\omega_1}$.
Many weak $P$-sets

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- We let every node in the Aronszajn tree that we used before correspond to a ‘compatible’ remote point in the Stone space of a certain subalgebra of $M_{\omega_1}$.
- A remote point of a space $X$ is a point $p \in \beta X \setminus X$ such that $p \notin \text{cl}_{\beta X} A$, for any nowhere dense subset $A$ of $X$.
Many weak $P$-sets

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- So a remote point of $X$ cannot be ‘reached’ by any nowhere dense subset of $X$. 
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- So a remote point of \( X \) cannot be ‘reached’ by any nowhere dense subset of \( X \).
- So in the Stone space of our measure algebra, such a point cannot be ‘reached’ by any countable subset of its complement.
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- It is known by the work of van Douwen and Chae and Smith that any nonspeudocompact space of countable $\pi$-weight has a remote point.
- We cannot apply that result, but in the case of measure algebras there is an easy way out.
To see this, let $X$ be any compact space, $\lambda$ a Radon probability measure on $X$, with the property that $\lambda(A) = 0$ for any nowhere dense $A \subseteq X$. We claim that $\mathbb{N} \times X$ has a remote point.
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\[ \lambda(F_i) \geq 1 - 2^{-i} \quad \cdots \quad \lambda(F_j) \geq 1 - 2^{-3} \quad \cdots \]

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Now we do! *Never forget a good result!*
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These are the main ingredients for the (quite involved) proof of the theorem.
Theorem (Dow and vM (2020))

There is a copy $X$ of $\mathbb{N}^*$ in $\mathbb{N}^*$ having the following properties:

1. There is a countable subset $E$ contained in $\mathbb{N}^* \setminus X$ such that the closure of $E$ contains $X$,

2. for every countable discrete subset $F$ in $\mathbb{N}^* \setminus X$, the closure of $F$ misses $X$. 

Klaas Pieter Hart and myself just completed an update on, and expansion of, our paper Open problems on $\beta\omega$ in the book Open Problems in Topology. See https://arxiv.org/abs/2205.11204. We invite comments, corrections, more problems, ...
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Many weak $P$-sets

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THANK YOU!