On a new problem of complexity theory arising from Galois-Tukey connections

(and a descriptive set theoretic representation)

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GT=Galois-Tukey connections/reductions

$\mathcal{M}$ – meager, $\mathcal{L}$ – Lebesgue null sets,

Luzin, Sierpiński, $\exists M \in \mathcal{M}$, $\exists L \in \mathcal{L}$, $M \cap L = \emptyset$, $M \cup L = [0, 1]$

$[0,1]$ _$\in$_ $\mathcal{L}$ x, $y \in [0, 1]$, $f(x) = x + M$, $g(y) = y - L$

f↓ g↑ st. $f(x) \not\ni y \Rightarrow x \in g(y)$, $\mathrm{Cov}(\mathcal{M}) \leq \mathrm{Non}(\mathcal{L})$

$\mathcal{M}$ _$\not\ni$_ $[0,1]$ and $\mathrm{Cov}(\mathcal{L}) \leq \mathrm{Non}(\mathcal{M})$ (*Rothberger*)

Consider **problems** $\mathbf{P_1} = (I_1, S_1, P_1)$, $P_1 \subseteq I_1 x S_1$, $\mathbf{P_2} = (I_2, S_2, P_2)$, $P_2 \subseteq I_2 x S_2$,

I_j – *instances*, S_j – *solution candidates*, $(i_j, s_j) \in P_j$ - s_j is a solution of i_j

I_1 _P_1_ S_1 A Galois-Tukey connection/reduction from $\mathbf{P_1}$ to $\mathbf{P_2}$

f↓ g↑ consists of a pair of mappings $f: I_1 \rightarrow I_2$, $g: S_2 \rightarrow S_1$

I_2 _P_2_ S_2 with $(\forall\, i_1 \in I_1)\,(\forall\, s_2 \in S_2)(\, f(i_1)\, P_2\, s_2 \Rightarrow i_1\, P_1\, g(s_2)\,)$

It follows $\mathfrak{b}(P_2) \leq \mathfrak{b}(P_1)$, $\mathfrak{d}(P_1) \leq \mathfrak{d}(P_2)$, P_2-solution gives a P_1-solution

J.W. Tukey. Convergence and Uniformity in Topology. Ann. Math. Studies Number 2, Princeton Univ. Press 1940

T. Bartoszynski. Additivity of measure implies additivity of category. Transactions of the American Mathematical Society 1984

F. Rothberger Eine Äquivalenz zwischen der Kontinuum Hypothese und Lusinschen und Sierpińskischen… Fund. Math. 30 (1938), 215–217

Vojtáš, P., Generalized Galois-Tukey connections … Israel Math. Conf. Proc. 6, American Mathematical Society, 1993, pp. 619–643

GT as reduction in complexity? Carefully …

GT 3SAT to 3Color

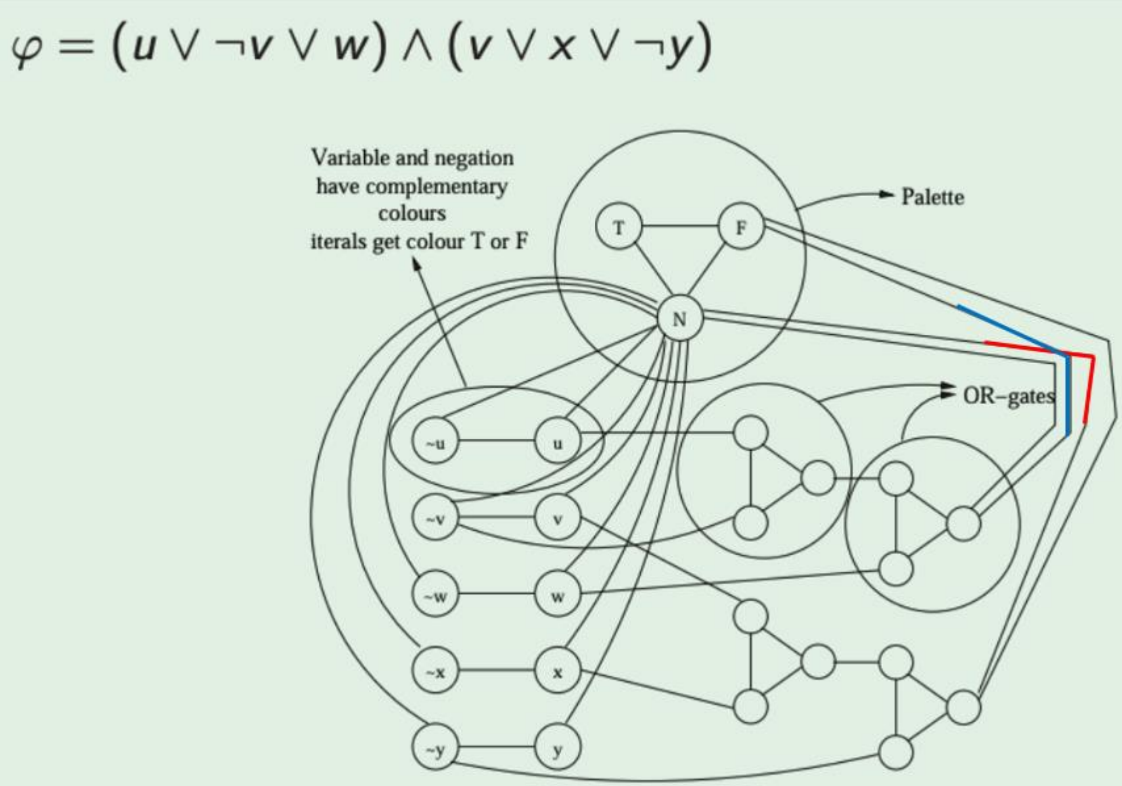
c 3-color of $G_\varphi \Rightarrow$

$\Rightarrow v_c$ makes φ true

$\varphi \in$ 3CNF _true in_ v_c:Var $\to 2$

reduction translation

G_φ _3-colorable by_ c:E$\to$3



but

there is a problem

“false $\Rightarrow$ * ” is true

send φ to a non-colorable graph makes GT work void

Blass’ first aid

$P^B = (I \cap dom(P), S, P)$

$3SAT^B$, $3COLOR^B$

Let’s $\to_{GT}$, $\leq_{GT}$ denote GT reduction

A. Blass. Questions and Answers -- A Category Arising in Linear Logic, Complexity Theory, and Set Theory (Advances in Linear Logic (ed. J.-Y. Girard, Y. Lafont, and L. Regnier) London Math. Soc. Lecture Notes 222 (1995) 61-81)

Complexity theory requires non-void reductions

Together with

c 3-color of $G_\varphi \Rightarrow$

$\Rightarrow v_c$ makes φ true

$\varphi \in$ 3CNF _true in_ v_c:Var $\to 2$

reduction translation

G_φ _3-colorable by_ c:E$\to 3$

complexity theory requires

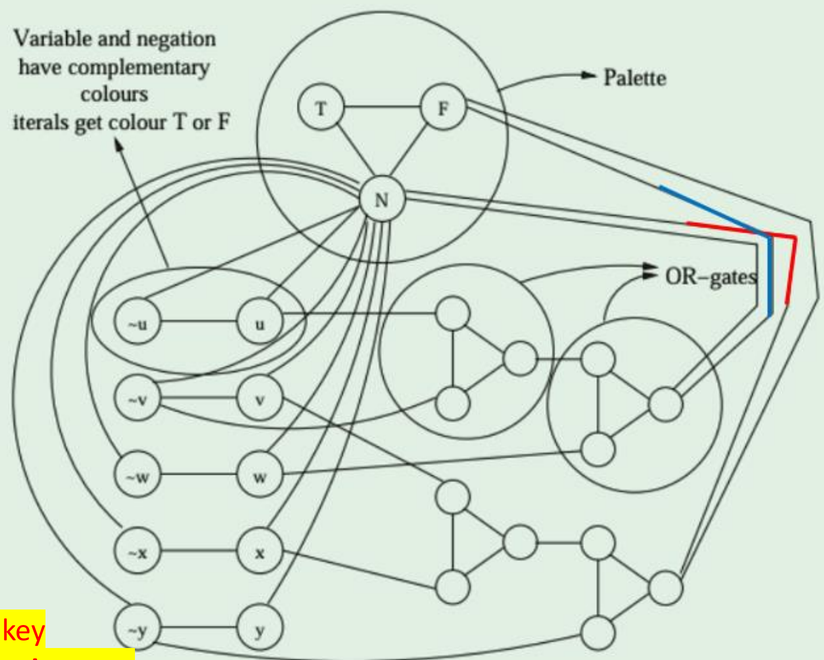
$(\exists v$:Var $\to 2)$

$(\varphi \in$ 3CNF true in v$)$

$\Rightarrow$ $(\exists$ c:E$\to 3)$

$(G_\varphi$ 3-colorable by c$)$

$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$



Let's $\to_{cth}$, $\leq_{cth}$ denote such reductions

Our original Galois-Tukey
Blass term Questions – Answers
We call it also Challenge – Response
Here problems, Instances - Solutions

Extended GT can give correct complexity reduction

Let us have a problem **P** = (I,S,P).

Put $S^n = S \cup$ **{no}** solutions with an extra element "no" = "no acceptable solution".
$P^n = S \cup \{(c, no): c \in C \setminus dom(P)\}$

$\mathbf{P_1^n} = (I_1, S_1^n, P_1^n)$, $\mathbf{P_2^n} = (I_2, S_2^n, P_2^n)$
(f,g) a GT from $\mathbf{P_1}$ to $\mathbf{P_2}$ extend to (f, g^n)
$g^n(no_2) = no_1$ and $(\forall i_1 \in I_1)(\forall s_2 \in S_2^n)$

$$P_2^n(f(i_1), s_2) \Rightarrow P_1(i_1, g^n(s_2))$$

$$P_2^n(f(i_1), no_2) \Rightarrow P_1^n(i_1, g^n(no_2))$$

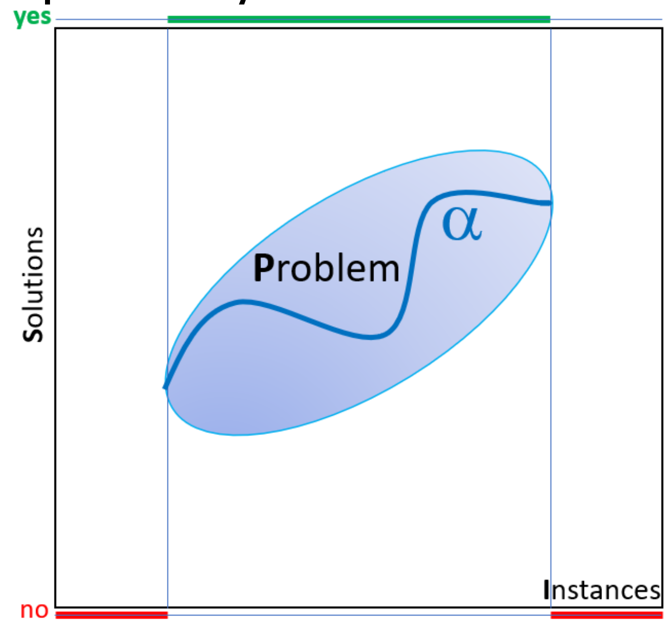
$$P_2^n(f(i_1), no_2) \Rightarrow P_1^n(i_1, no_1)$$

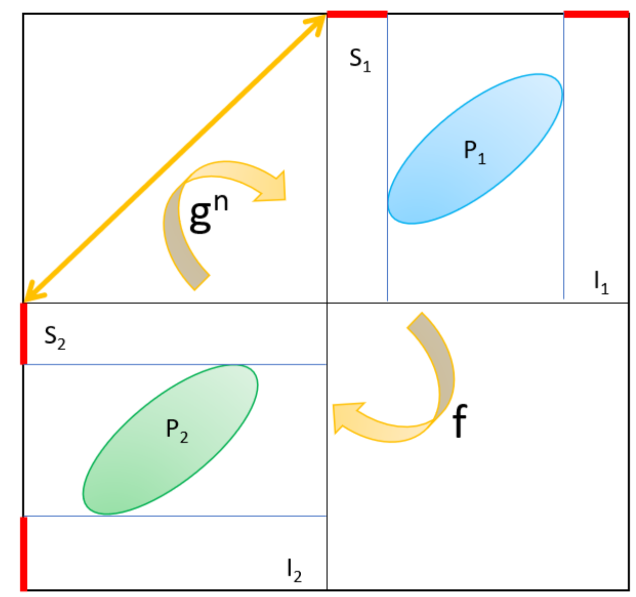
$$\neg P_2^n(f(i_1), no_2) \Leftarrow \neg P_1^n(i_1, no_1)$$

$$\exists s_2\, P_2^n(f(i_1), s_2) \Leftarrow \exists s_1\, P_1^n(i_1, s_1)$$

This is the complexity theory requirement hence

$$\mathbf{P_1^n} \rightarrow_{GT} \mathbf{P_2^n} \text{ implies } \mathbf{P_1} \rightarrow_{cth} \mathbf{P_2}$$





Several classes of problems, reductions

Reductions – classical $\to_{cth}$, $\leq_{cth}$ and GT $\to_{GT}$, $\leq_{GT}$,
Problems – classical 3SAT, 3COLOR, New-$3SAT^n$, $3COLOR^n$
New class $\mathbf{C}^n$, in PSPACE, **wisdom** of both NP and coNP

Problems

Search problem **P**=(I,S,P)

polynomialy bounded, if

exists a polynomial q

$$(i,s)\in P \text{ iff } |s|\leq q(|i|)$$

Decision problem $\mathbf{P}^d$ =
=(I,S^d,P^d), S^d={0-no,1-yes}

$$P^d(i,1) \text{ iff } (\exists s)(P(I,s))$$

Solutions

P is *efficiently solvable* by polynomial time alg A s.t.

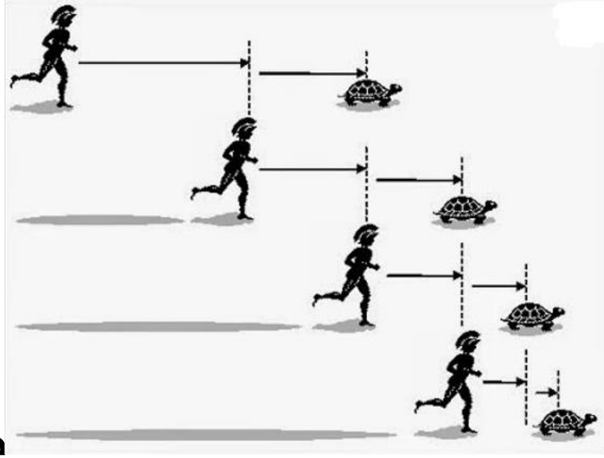
$$A(x)\in P(x) \text{ iff } P(x)\neq\emptyset$$

$$P(x)=\emptyset \text{ then } A(x) = \bot$$

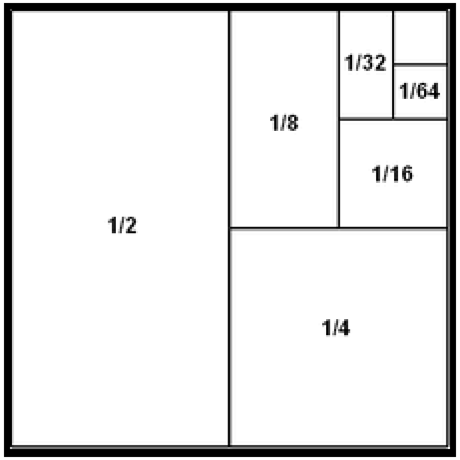
P - *efficiently checkable solutions* - **p**oly**t**ime**a**lg A

$$A(i, s) \text{ iff } (I, s) \in P$$

Zeno's paradox and computers



- Time complexity a little bit like Achilles and tortoise
- Calculate $\sum_{i=1}^{\infty} \frac{1}{2^i}$ by a Turing machine
    - Computable analysis
- It is the problem of infinity (actual)
- Geometrically?



- Infinity in the weakest possible theory?
    - Reverse mathematics (weak 2^{nd} order PA)
- Infinity in the strongest possible theory?
    - Descriptive set theory (to have tools for quality assessment) – this will be our approach here

Computable analysis, reverse mathematics New problems of complexity theory

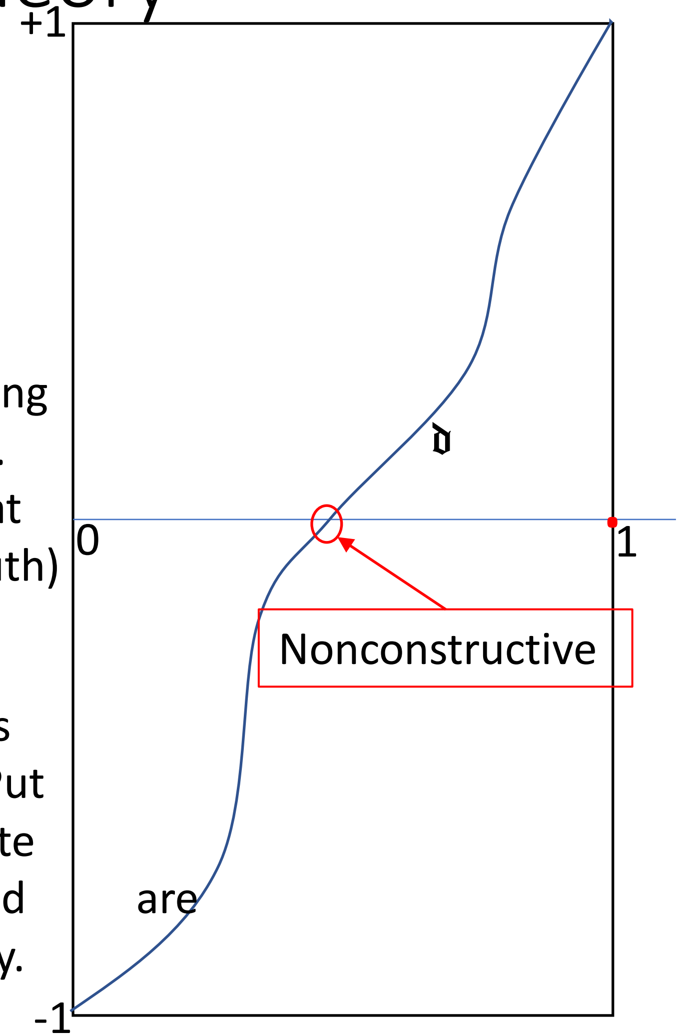
Computable analysis - CA

A **real** $x = 0, x_1 \ldots x_n \ldots$ is **computable** if there is a Turing machine T_x such that $T_x(n) = x_n$

A **function** f on reals is **computable** if there is a Turing machine T_f such that $T_f(T_x) = T_{f(x)}$

A function f is **constructively continuous** if there is a Turing machine T_c such that for every $T_\varepsilon > 0$ $T_c(T_\varepsilon) = T_\delta$ works for T_f.

There is a constructively continuous function $\mathfrak{d}$ such that $\mathfrak{d}(0) = -1$, $\mathfrak{d}(1) = +1$ and **there is no** real x with $\mathfrak{d}(x) = 0$ (Demuth)



Reverse mathematics - RM

studies strength of statements $(\forall X)(\exists Y)\varphi(X, Y)$ in various weak versions of 2nd order Peano arithmetic WKL, RT^m_k … Put $P_\varphi = \{(X,Y) \mid \varphi(X, Y)\}$, but domain and range consist of infinite sets. Our problems have instances and solutions **finite** and are of type $(\forall x)(\exists y)\varphi(x, y)$ but some requirements to x, y apply. In RM, ask for **quality** of infinite subtrees, of $2^{<\omega}$ infinite paths, **quality** of $f : [\omega]^n \to k$ and infinite homogeneous sets *H, …*

All we need is some sort of coding problems to $[0,1]^2$

Coding finite objects to finite binary reals

$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$ $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_4 \vee \neg x_5) \in 3CNF^5$

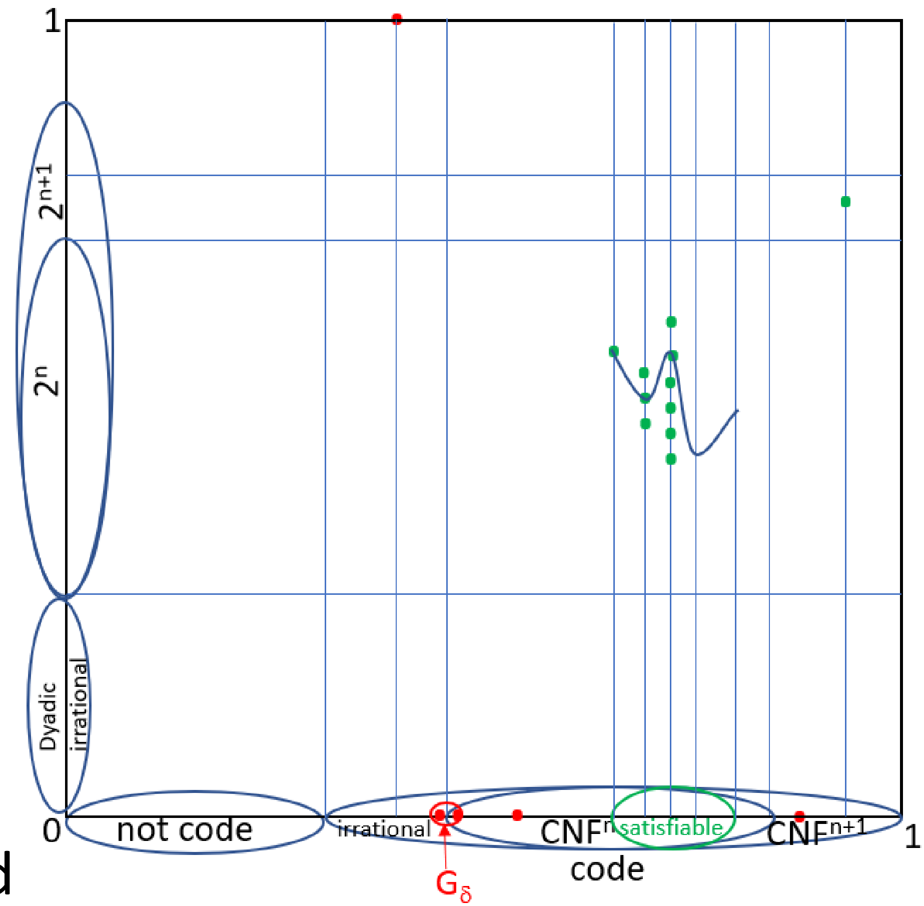
Clause – conjunctions of ()()(), need not to code $\wedge$

Literals – disjunctions of [][][], need not to code $\vee$

A literal [“maybe$\neg$” “variable identified by index-number”]

(coded as 000	([$x_1 \approx (1)_{10} \approx (1)_2$] [$\neg$
) coded as 001	000 010 111 011 010 101
[coded as 010	
] coded as 011	$x_2 \approx (2)_{10} \approx (10)_2$ $x_3 \approx (3)_{10} \approx (11)_2$])
□ coded as 100	111 110 111 111 011 001
$\neg$ coded as 101	$(x_1 \vee \neg x_2 \vee x_3)$ is coded as
0 coded as 110	0,000010111011010101111110111111011001
1 coded as 111	code $\varphi = \ulcorner \varphi \urcorner = 0{,}x_1^{\varphi} \dots x_i^{\varphi} \dots x_{l-3}^{\varphi}001$

Multivalued satisfaction function $\mathfrak{s}$ on $\ulcorner\varphi\urcorner$



Satisfiable $\mathfrak{s}(\ulcorner\varphi\urcorner) \subseteq 2^{var(\varphi)}$ …

Unsatisfiable $\mathfrak{s}(\ulcorner\varphi\urcorner) = 0$

$\ulcorner\varphi\urcorner \sqsubseteq \ulcorner\psi\urcorner$ binary prolongation

(codes, $\sqsubseteq$) form an infinite tree

branches - dyadic irrationals $x \in I$

$\mathfrak{s}(x) = \underline{\lim}\, sgn(\mathfrak{s}(x{\downarrow}n_i))$, where

$x{\downarrow}n_i = \ulcorner\varphi_i\urcorner$ and $\ulcorner\varphi_i\urcorner \sqsubseteq \ulcorner\varphi_{i+1}\urcorner$

$\mathfrak{s}(\ulcorner\varphi\urcorner)=0, \ulcorner\varphi\urcorner \sqsubseteq \ulcorner\psi\urcorner$ then $\mathfrak{s}(\ulcorner\psi\urcorner)=0$

$B(\varphi)=\{\ulcorner\psi\urcorner: \ulcorner\varphi\urcorner \sqsubseteq \ulcorner\psi\urcorner\}$ open …

$b(\varphi)=\{x \in I, \exists i\ x{\downarrow}n_i=\ulcorner\varphi\urcorner\}$ is G_δ

$\mathfrak{s}(\ulcorner\varphi\urcorner)=0$ then $\mathfrak{s}(B(\varphi)\cup b(\varphi)) = 0$ and

$\mathfrak{s}$ is **upper semicontinuous** on $B(\varphi)\cup b(\varphi)$

$\mathfrak{s}(\ulcorner\varphi\urcorner) \neq \emptyset$ then $(\exists x \in I)(\ulcorner\varphi\urcorner \sqsubseteq x$ and $\mathfrak{s}(x)=1)$

Estimate quality of $\mathfrak{s} \subseteq [0,1]x[0,1]$ and quality of selectors

Reduction of ⌈3SAT⌉ to ⌈3COLOR⌉

$x \in I$, $x{\downarrow}n_i = \lceil \varphi_i \rceil$ and $\lceil \varphi_i \rceil \sqsubseteq \lceil \varphi_{i+1} \rceil$, $\lceil \varphi_i \rceil \to x$ classically

$y_i \to x$, then $\forall l\ \exists l_0 \geq l\ \exists i_0\ \forall i \geq i_0\ y_i{\downarrow}l_0 = x{\downarrow}l_0$ is a code

y_i from l_0 on can be seen as disturbance

$\varphi = \lceil \varphi \rceil = 0{,}x_1^{\varphi}\ldots x_i^{\varphi}\ldots x_{l-3}^{\varphi}001$

$B(\varphi) = \{\lceil \psi \rceil : \lceil \varphi \rceil \sqsubseteq \lceil \psi \rceil\} \subseteq [(\lceil \varphi \rceil{\downarrow}l\text{-}3)001, (\lceil \varphi \rceil{\downarrow}l\text{-}3)01]$

Graphs G=(V,E) can be **coded** similarly(fix V ordering)

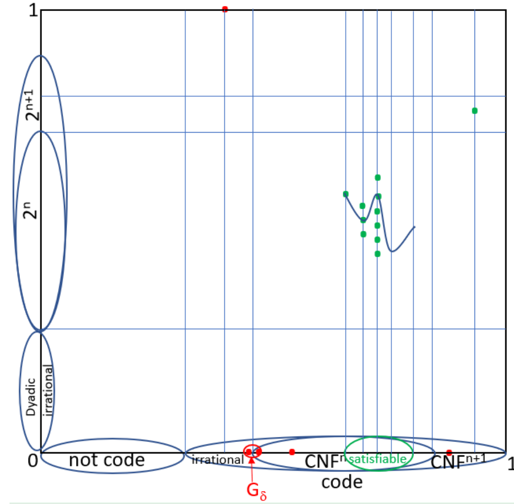
(Un)3vertex-corolable $\mathfrak{s}_{\mathfrak{G}}(\lceil G \rceil) = 0$ or in $3^{|V|}$ similarly

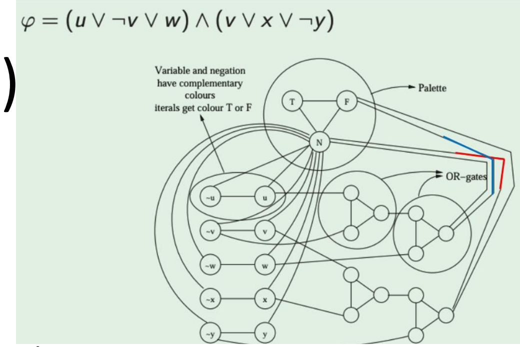
codes ⌈G⌉ ordered by ⊑ form an infinite tree

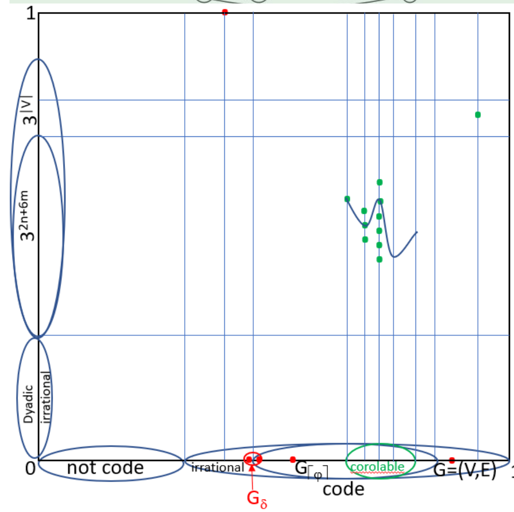
Reduction is continuous – additional clause just ads additional vertices and some edges

Interpretation of coloring is continuous – just read colors “T”/”F” of vertices coding variables (mod 2) in the same order as variables

We have a **continuous GT** from ⌈3SAT⌉ to ⌈3COLOR⌉







Quality of problems and solutions

Problems- 3SAT…,$3SAT^d$…, $3SAT^n$… in PA,$[0,1]^2$
Solutions - poly**a**lgorithms, topological selectors
Reductions - poly**a**lgorithms, continuous reductions
Classes – $\mathbb{P}$, $\mathbb{NP}$, …. $\mathbb{P}^B$, $\mathbb{NP}^B$,…. $\mathbb{P}^n$, $\mathbb{NP}^n$ in PA / $[0,1]^2$

Problems	Solutions
Search problem **P**=(I,S,P) *polynomially bounded* *Decision problem* $\mathbf{P}^d$ P $\subseteq$ [0,1]x[0,1] quality as - multivalued function - as subset of [0,1]x[0,1]	**P** is *efficiently solvable* **P** - *efficiently checkable solutions* (*$\mathbf{P}^n$ is not*) Quality of selectors -partial / total - continuous? Borel? …

Thank you

Questions, Comments?