Scattered P-spaces of weight ω_1

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Introduction

The name "P-space" was used in 1954 by L. Gillman and M. Henriksen as an abbreviation for "pseudo-discrete space", which means a completely regular space X such that for every point x in X, every continuous function $X \to \mathbb{R}$ is constant on some neighbourhood of x.

Proposition

A completely regular space X is pseudo-discrete if and only if the intersection of a countable family of open subsets of X is open.

In 1972 A. K. Misra proposed consideration of T_1-spaces satisfying this condition: *the intersection of a countable family of open subsets is open* and called them *P-spaces*.

- Following M. Fréchet, if a space X can be embedded into a space Y, then we write $X \subset_h Y$.
- If $X \subset_h Y$ and $Y \subset_h X$, then we write $X =_h Y$ and say that X and Y have the same *topological rank* (K. Kuratowski) or *dimensional type* (W. Sierpiński).

Theorem (W. D. Gillam, 2005)

The quasi-ordered set $(\mathcal{P}(\mathbb{Q}), \subset_h)$ contains neither an infinite antichain, nor an infinite descending chain.

- A topological space is scattered if its every non-empty subspace contains an isolated point.
- If X is a topological space and α is an ordinal number, then $X^{(\alpha)}$ denotes the α-th derivative of X.
- If X is a scattered space, then Cantor–Bendixson rank of X is the least ordinal $N(X)$ such that the derivative $X^{(N(X))}$ is empty.
- If X is a scattered P-space of cardinality ω_1, then $N(X) < \omega_2$.

Proposition

A scattered space of weight at most ω_1 is of cardinality at most ω_1.

P-base at a point

- A point of a regular P-space of weight ω_1 has a base $\{V_\alpha : \alpha < \omega_1\}$ with the following properties:
  - $V_0 = X$ and sets V_α are closed-open,
  - $V_\beta \supseteq V_\alpha$ for $\beta < \alpha < \omega_1$,
  - $V_\alpha = \bigcap \{V_\beta : \beta < \alpha\}$ for a limit ordinal number $\alpha < \omega_1$.
- We will say that $\{V_\alpha : \alpha < \omega_1\}$ is a P-base.
- The sets $V_\alpha \setminus V_{\alpha+1}$ will be called slices.
- There are exactly three, up to homeomorphism, P-spaces with one accumulation point: the number of uncountable slices can be 0, 1 or ω_1, which corresponds to spaces denoted by $i(2)$, $i(2) \oplus D$ and $J(2)$, respectively, where D is a discrete space of cardinality ω_1.

Spaces $J(\alpha)$

- If $J(\alpha)$ is defined, then we assume that $J(\alpha+1)$ is a P-space with a P-base at a point $x \in J(\alpha+1)$ such that its slices are homeomorphic to the sum $\bigoplus_{\omega_1} J(\alpha)$.
- If $\beta < \omega_2$ is a limit ordinal, then we define $J(\beta) = \bigoplus_{\alpha<\beta} J(\alpha)$.

Proposition

- *If X is an elementary P-space of weight ω_1 with $N(X) = n < \omega$, then $X \subset_h J(n)$.*
- *If X is a scattered P-space of weight ω_1 such that $N(X) > 2n$, then $J(n+1) \subset_h X$.*

The assumption $N(X) > 2n$ is optimal in the sense that spaces $i(4)$ and $J(3)$ have incomparable dimensional types.

Spaces $i(2), i(3), \ldots$

If $n < \omega$ and the space $i(n-1)$ has been defined, we assume that the space $i(n)$ has the following properties:

- $i(n)^{(n-1)} = \{x\}$ (hence $N(i(n)) = n$),
- there is a P-base $\{V_\alpha \colon \alpha < \omega_1\}$ at x such that each slice $V_\alpha \setminus V_{\alpha+1}$ is homeomorphic to $\bigoplus_\omega i(n-1)$.

Embedding scattered P-spaces into ordinals

- B. Knaster and K. Urbanik showed that a separable scattered and metrizable space can be embedded in a sufficiently large countable ordinal number.
- R. Telgársky removed the assumption of separability from the above result.

Theorem

Any scattered P-space of weight ω_1 can be embedded into ω_2.

Elementary sets

A closed-open subset E of a P-space is elementary, whenever $N(E)$ is a successor ordinal and $E^{(N(E)-1)}$ is a singleton.

Lemma

If X is a scattered P-space of weight ω_1, then any open cover of X can be refined by a partition consisting of elementary sets.

- Spaces $i(2)$ and $J(2)$ have the following property of being stable: in both cases, there is a P-base at the accumulation point such that its slices are pairwise homeomorphic.
- Although the space $i(2) \oplus D$ has not such a P-base, its accumulation point has a neighbourhood U such that U is stable.
- A stable P-space is determined by a single slice of its P-base.

Proposition

For each $n < \omega$, there exist only finitely many non-homeomorphic stable sets with with rank n. Any elementary set with finite rank is the sum of a family of stable sets.

Proof.

- If $\{V_\alpha \colon \alpha < \omega_1\}$ is a P-base at x in elementary set E with rank n, then $\{V_\alpha \setminus V_{\alpha+1} \colon \alpha < \omega_1\}$ is a cover of $E \setminus \{x\}$.
- There is a partition $\mathcal{R}$, which refines this cover and consists of elementary sets.
- Each slice $V_\alpha \setminus V_{\alpha+1}$ is the sum of stable sets with rank lower than n.
- Each stable set with rank lower than n appears in a slice countably or uncountably many times and we can make it uniform beginning at some $\alpha < \omega_1$.
- We can also assure that each stable set appears 0-, ω- or ω_1-many times in a slice $V_\gamma \setminus V_{\gamma+1}$, for $\gamma > \alpha$.

Theorem

Let $(\mathcal{F}, \subset_h)$ be an ordered set, where $\mathcal{F}$ is a family of scattered P-spaces of weight ω_1 with ranks $\leq n$.
Then every antichain is finite and every strictly decreasing chain is finite.

Proof.

- Let $F_1, \ldots, F_m$ be all stable sets with rank not greater than n.
- We know that a scattered P-space X can be partitioned into stable sets, hence $X = \bigoplus_{\kappa_1^X} F_1 \oplus \ldots \oplus \bigoplus_{\kappa_m^X} F_m$, where $\kappa_i^X \leqslant \omega_1$.
- Thus we have defined a function $X \mapsto \varphi(X) = (\kappa_1^X, \ldots, \kappa_m^X) \in [0, \omega_1]^m$.
- Observe that:
  (1) $\varphi(X) \leqslant \varphi(Y) \Rightarrow X \subset_h Y$;
  (2) $X \subset_h Y$ and $Y \not\subset_h X \Rightarrow \exists_i \kappa_i^X < \kappa_i^Y$.
- This takes us to the coordinate-wise ordered set $[0, \omega_1]^m \ldots$

. . . where we can use the following variant of Bolzano–Weierstrass theorem.

Proposition

If P is a well-ordered set, then any infinite subset of coordinate-wise ordered P^m contains an infinite increasing sequence.
In particular, any antichain and any decreasing sequence in P^m are finite. □

Corollary

Let $(\mathcal{F}, \subset_h)$ be an ordered set, where $\mathcal{F}$ is a family of scattered P-spaces of weight ω_1 with finite Cantor–Bendixson rank. Then every antichain, every strictly decreasing chain are finite.

Proof.

- Fix an antichain $\mathcal{A} \subseteq \mathcal{F}$.
- If $X, Y \in \mathcal{A}$, then $2N(X) < N(Y)$ is impossible, otherwise $X \subset_h J(N(X)) \subset_h Y$.
- Thus spaces in $\mathcal{A}$ have ranks bounded by some $n < \omega$.
- Suppose $X_1 \supset_h X_2 \supset_h \dots$ is a strictly decreasing sequence of scattered P-spaces with finite rank.
- Then all spaces X_n have ranks not greater than $N(X_1)$.

□

Scattered P-spaces with infinite rank

Proposition

If X is a scattered P-space such that $N(X) = \omega$, then $X =_h J(\omega)$.

Proof.

- Let $X = \bigcup\{E_\gamma : \gamma < \omega_1\}$ be a partition such that each E_γ is an elementary set.
- For each n there exists γ such that $N(E_\gamma) > 2n$, hence $J(n) \subset_h E_\gamma$.
- Therefore $J(\omega) \subset_h X$.
- Since $N(E_\gamma) < \omega$, we have $E_\gamma \subset_h J(\omega)$.
- Because of $J(\omega) \cong \bigoplus_{\omega_1} J(\omega)$, we get $X \subset_h J(\omega)$.

□

Scattered P-spaces with infinite rank

Proposition

If Y is an elementary set of weight ω_1 with Cantor–Bendixson rank $\alpha+1$, then $Y \subset_h J(\alpha+1)$.

Corollary

If X and Y are elementary sets with Cantor–Bendixson rank $\omega+1$, both of the weight ω_1, then $X =_h Y$.

Proposition

If $\beta \in Lim$, then $X \subset_h J(\beta)$ for each P-space X with $N(X) \leqslant \beta$.

Theorem

If X is a scattered P-space of weight ω_1 and Y is a crowded P-space of weight ω_1, then $X \subset_h Y$.

Scattered P-spaces with infinite rank

Theorem

If $\beta < \omega_1$ is limit ordinal, then $J(\beta) =_h X$ for each P-space X with $N(X) = \beta$, and also $J(\beta + n) \subset_h Z$ for each elementary set with $N(Z) = \beta + 2n - 1$ for $n > 0$.

Corollary

If $\beta \in Lim$ and X is an elementary set with $N(X) = \beta + 1$, then $J(\beta + 1) =_h X$. □

Theorem

If $\beta < \omega_1$ is a limit ordinal, then the classes of regular P-spaces

$$\{X : 0 < N(X) \leqslant \omega + 1\} \text{ and } \{X : \beta < N(X) \leqslant \beta + \omega + 1\}$$

are $\subset_h$-isomorphic.

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