Almost disjoint families and relative versions of covering properties of κ-paracompactness type

Samuel G. da Silva

UFBA, Salvador/Bahia/Brazil
(travel sponsored by FAPESB)

TOPOSYM 2016 – Prague, Czech Republic

This is a joint work with Charles Morgan (UCL, London) and Dimi Rangel (USP, Sao Paulo).

25–29 July, 2016

Dedicatory: Ofelia Alas and Richard Wilson

This paper is an enlarged, revised and improved version of a poster presented by Dimi Rangel at STW 2013 (the event honouring the 70th anniversary of Ofelia Alas – Maresias, Brazil), and it was accepted for publication in the proceedings of MICTA 2014 (the event honouring the 70th anniversary of Richard Wilson – Cocoyoc, México). Nevertheless, this is the first oral presentation of this work.

The authors are very happy to dedicate this work to both professors Ofelia T. Alas and Richard G. Wilson.

The speaker acknowledges Frank Tall by calling his attention to Zenor's property $\mathcal{B}$ during the 2012 and 2013 editions of STW.

ψ-spaces

We assume the audience is very familiar with **Isbell–Mrówka spaces** (or Ψ-spaces), which are spaces constructed from almost disjoint families of infinite sets of ω (under a standard, well-known construction).

Such spaces were introduced in the 50's (Mrówka, Katětov,...) and constitute, since then, a fruitful source of examples and counterexamples.

It is very usual that topological properties of a given Ψ-space may be combinatorially characterized in terms of the almost disjoint family used in the construction.

Countable paracompactness of Ψ-spaces

Combinatorial characterization of countable paracompactness (M., da S. – 09)

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an a.d. family and consider $\Psi(\mathcal{A})$. TFAE:

(i) $\Psi(\mathcal{A})$ is countably paracompact.

(ii) For every decreasing sequence $\langle \mathcal{F}_n : n < \omega \rangle$ of subsets of $\mathcal{A}$ such that $\bigcap_{n<\omega} \mathcal{F}_n = \emptyset$ there is a sequence $\langle E_n : n < \omega \rangle$ of subsets of ω satisfying the conditions:

(ii).1 $\forall n < \omega \ \forall A \in \mathcal{F}_n \ (A \setminus E_n \text{ is finite})$; and
(ii).2 $\forall A \in \mathcal{A} \ \exists n < \omega \ (A \cap E_n \text{ is finite})$.

(iii) For every function $g : \mathcal{A} \to \omega$ there are a $\subseteq$-decreasing sequence $\langle E_n : n < \omega \rangle$ of subsets of ω and a function $f : \mathcal{A} \to \omega$ satisfying the conditions:

(iii).1 $\forall A \in \mathcal{A} \ (A \setminus E_{g(A)} \text{ is finite})$; and
(iii).2 $\forall A \in \mathcal{A} \ (A \cap E_{f(A)} \text{ is finite})$.

Towards a relative definition

The item (*ii*) of the preceding slide resembles the well-known Ishikawa's characterization of countable paracompactness in terms of decreasing sequences of closed sets with empty intersection.

In a sense, it has shown that, for Ψ-spaces, the only decreasing-with-empty-intersection sequences of closed subsets that matter are those from subsets of the almost disjoint family itself.

Only a few years later the speaker realized that this also had the smell of **relative topological properties**. Let us go in this direction; some terminology ...

If $Y \subseteq X$, we will say that $\mathcal{V}$ is **locally finite at** Y if it is locally finite at every point of Y, meaning that every $y \in Y$ has a neighbourhood which intersects at most finite elements of $\mathcal{V}$.

Analogously, given any uncountable cardinal κ, one can define the notion of a family being **locally smaller than** κ **at** Y.

Relative countable paracompactness

The following notion was introduced by the speaker in 2007:

da S., 07 – Relatively countably paracompact spaces

Let X be a topological space and $Y \subseteq X$. We say that Y is **relatively countably paracompact in** X if for every countable open cover $\mathcal{U}$ of X there is a family of open sets $\mathcal{V}$ such that $\mathcal{V}$ refines $\mathcal{U}$, $\mathcal{V}$ is locally finite at Y and $Y \subseteq \bigcup \mathcal{V}$.

We have shown in 2007 (using well-known results on **dominating families** in ${}^{\omega_1}\omega$) that: the existence of a separable space X with an uncountable closed discrete subset which is relatively countably paracompact in X cannot be proved within **ZFC** – since, under certain assumptions, it would imply the existence of inner models with measurable cardinals. The relationship between countable paracompactness, separability of spaces with uncountable closed discrete subsets and dominating families was first noticed by Watson in 1985.

Equivalences of relative countable paracompactness

In the present work, we have returned to this relative topological property.

First, let us characterize it.

Some characterizations (general case) – M., R., da S. 2015

Let X be a topological space and $Y \subseteq X$. The following statements are equivalent:

(i) Y is relatively countably paracompact in X;

(ii) For every open cover $\mathcal{U} = \{U_i : i < \omega\}$ of X there is a family of open sets $\mathcal{V} = \{V_i : i < \omega\}$ satisfying $V_i \subseteq U_i$ for each $i < \omega$ and such that $\mathcal{V}$ is locally finite in Y and $Y \subseteq \bigcup \mathcal{V}$;

Equivalences of relative countable paracompactness

Conditions on increasing open covers and on decreasing sequences of closed sets with empty intersection

(*iii*) For every decreasing sequence $\langle C_i : i < \omega \rangle$ of closed subsets of X with $\bigcap_{i<\omega} C_i = \emptyset$ there is a sequence $\langle A_i : i < \omega \rangle$ of open subsets of X satisfying $C_i \cap Y \subseteq A_i$ for each $i < \omega$ and such that $\bigcap_{i<\omega} \overline{A_i} \cap Y = \emptyset$;

(*iv*) For every increasing open cover $\{O_i : i < \omega\}$ of X there is a sequence $\langle G_i : i < \omega \rangle$ of closed subsets of X satisfying $G_i \cap Y \subseteq O_i$ for each $i < \omega$ and such that $Y \subseteq \bigcup_{i<\omega} \mathrm{int}(G_i)$.

A.d. families which are **relatively** countably paracompact !

Comparing both characterizations, one can conclude, as the speaker did in 2011, that $\Psi(\mathcal{A})$ **is countably paracompact, if, and only if,** $\mathcal{A}$ **is relatively countably paracompact in** $\Psi(\mathcal{A})$.

This fact lead the authors to believe that the natural way (from both topological and combinatorial points of view) of studying covering properties of κ-paracompactness type for Isbell–Mrówka spaces (looking for possible uncountable generalizations/versions) will be by investigating the conditions under which a given almost disjoint family satisfies **relative** versions of these properties in its corresponding Ψ-space.

This is what will be done presently. Before that, we will show that MAD families are not countably paracompact.

MAD families are not countably paracompact

We got to one of the main results of this work . . . And, indeed, it was the starting point of this research.

If $\mathcal{A}$ is a MAD family, then $\mathcal{A}$ is not countably paracompact.

We, in fact, prove a stronger result – which is interesting *per se*. It should be clear that proving the following proposition suffices to ensure the validity of the previous statement – in view of (iii) of the combinatorial characterization of countable paracompactness in Ψ-spaces.

Proposition (Morgan, Rangel, da S. – 2015)

Suppose $\mathcal{A}$ is a MAD family of infinite subsets of ω and let $\langle E_n : n < \omega \rangle$ be a $\subseteq$-decreasing sequence of infinite subsets of ω. Under these assumptions, there is no function $f : \mathcal{A} \to \omega$ such that

$$\forall A \in \mathcal{A} \, (A \cap E_{f(A)} \text{ is finite}).$$

Sketch of the proof

- $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ MAD, $\langle E_n : n < \omega \rangle$ $\subseteq$-decreasing. Suppose, towards a contradiction, that $f : \kappa \to \omega$ is such that $A_\alpha \cap E_{f(\alpha)}$ is finite for every α.
- Using the hypothesis and maximality, we inductively construct a sequence $\langle \alpha_n : n < \omega \rangle$ of distinct ordinals in κ and a increasing sequence of naturals $\langle k_n : n < \omega \rangle$ such that, for every $n < \omega$, $A_{\alpha_n} \cap E_{k_n}$ is infinite but $A_{\alpha_n} \cap E_{k_{n+1}}$ is finite ("take $k_{n+1} = f(\alpha_n)$", etc.)
- Define the disjoint family $\{B_n : n < \omega\}$, $B_n = A_{\alpha_n} \setminus \bigcup_{m<n} A_{\alpha_m}$. Notice that each $B_n \cap E_{k_n}$ is infinite (and, therefore, non-empty !).
- Pick $x_n \in B_n \cap E_{k_n}$ and let C be the infinite set $C = \{x_n : n < \omega\}$.
- By maximality, there is some β such that $C \cap A_\beta$ is an infinite set, let $C \cap A_\beta = \{x_{n_i} : i < \omega\}$.
- Notice that if $n_i > n$ then $x_{n_i} \in E_{k_n}$ (the sequence of E_n's is decreasing !), and therefore $A_\beta \cap E_{k_n}$ is infinite for every $n < \omega$.
- As k_n's increase and E_n's decrease, it follows that $A_\beta \cap E_n$ is infinite for every $n < \omega$ – and this contradicts "$A_\beta \cap E_{f(\beta)}$ is finite". ■

Sketch of the proof

- $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ MAD, $\langle E_n : n < \omega \rangle$ $\subseteq$-decreasing. Suppose, towards a contradiction, that $f : \kappa \to \omega$ is such that $A_\alpha \cap E_{f(\alpha)}$ is finite for every α.
- Using the hypothesis and maximality, we inductively construct a sequence $\langle \alpha_n : n < \omega \rangle$ of distinct ordinals in κ and a increasing sequence of naturals $\langle k_n : n < \omega \rangle$ such that, for every $n < \omega$, $A_{\alpha_n} \cap E_{k_n}$ is infinite but $A_{\alpha_n} \cap E_{k_{n+1}}$ is finite ("take $k_{n+1} = f(\alpha_n)$", etc.)
- Define the disjoint family $\{B_n : n < \omega\}$, $B_n = A_{\alpha_n} \setminus \bigcup_{m<n} A_{\alpha_m}$. Notice that each $B_n \cap E_{k_n}$ is infinite (and, therefore, non-empty !).
- Pick $x_n \in B_n \cap E_{k_n}$ and let C be the infinite set $C = \{x_n : n < \omega\}$.
- By maximality, there is some β such that $C \cap A_\beta$ is an infinite set, let $C \cap A_\beta = \{x_{n_i} : i < \omega\}$.
- Notice that if $n_i > n$ then $x_{n_i} \in E_{k_n}$ (the sequence of E_n's is decreasing !), and therefore $A_\beta \cap E_{k_n}$ is infinite for every $n < \omega$.
- As k_n's increase and E_n's decrease, it follows that $A_\beta \cap E_n$ is infinite for every $n < \omega$ – and this contradicts "$A_\beta \cap E_{f(\beta)}$ is finite". ■

Sketch of the proof

- $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ MAD, $\langle E_n : n < \omega \rangle$ $\subseteq$-decreasing. Suppose, towards a contradiction, that $f : \kappa \to \omega$ is such that $A_\alpha \cap E_{f(\alpha)}$ is finite for every α.
- Using the hypothesis and maximality, we inductively construct a sequence $\langle \alpha_n : n < \omega \rangle$ of distinct ordinals in κ and a increasing sequence of naturals $\langle k_n : n < \omega \rangle$ such that, for every $n < \omega$, $A_{\alpha_n} \cap E_{k_n}$ is infinite but $A_{\alpha_n} \cap E_{k_{n+1}}$ is finite ("take $k_{n+1} = f(\alpha_n)$", etc.)
- Define the disjoint family $\{B_n : n < \omega\}$, $B_n = A_{\alpha_n} \setminus \bigcup_{m<n} A_{\alpha_m}$. Notice that each $B_n \cap E_{k_n}$ is infinite (and, therefore, non-empty !).
- Pick $x_n \in B_n \cap E_{k_n}$ and let C be the infinite set $C = \{x_n : n < \omega\}$.
- By maximality, there is some β such that $C \cap A_\beta$ is an infinite set, let $C \cap A_\beta = \{x_{n_i} : i < \omega\}$.
- Notice that if $n_i > n$ then $x_{n_i} \in E_{k_n}$ (the sequence of E_n's is decreasing !), and therefore $A_\beta \cap E_{k_n}$ is infinite for every $n < \omega$.
- As k_n's increase and E_n's decrease, it follows that $A_\beta \cap E_n$ is infinite for every $n < \omega$ – and this contradicts "$A_\beta \cap E_{f(\beta)}$ is finite". $\blacksquare$

Sketch of the proof

- $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ MAD, $\langle E_n : n < \omega \rangle$ $\subseteq$-decreasing. Suppose, towards a contradiction, that $f : \kappa \to \omega$ is such that $A_\alpha \cap E_{f(\alpha)}$ is finite for every α.
- Using the hypothesis and maximality, we inductively construct a sequence $\langle \alpha_n : n < \omega \rangle$ of distinct ordinals in κ and a increasing sequence of naturals $\langle k_n : n < \omega \rangle$ such that, for every $n < \omega$, $A_{\alpha_n} \cap E_{k_n}$ is infinite but $A_{\alpha_n} \cap E_{k_{n+1}}$ is finite ("take $k_{n+1} = f(\alpha_n)$", etc.)
- Define the disjoint family $\{B_n : n < \omega\}$, $B_n = A_{\alpha_n} \setminus \bigcup_{m<n} A_{\alpha_m}$. Notice that each $B_n \cap E_{k_n}$ is infinite (and, therefore, non-empty !).
- Pick $x_n \in B_n \cap E_{k_n}$ and let C be the infinite set $C = \{x_n : n < \omega\}$.
- By maximality, there is some β such that $C \cap A_\beta$ is an infinite set, let $C \cap A_\beta = \{x_{n_i} : i < \omega\}$.
- Notice that if $n_i > n$ then $x_{n_i} \in E_{k_n}$ (the sequence of E_n's is decreasing !), and therefore $A_\beta \cap E_{k_n}$ is infinite for every $n < \omega$.
- As k_n's increase and E_n's decrease, it follows that $A_\beta \cap E_n$ is infinite for every $n < \omega$ – and this contradicts "$A_\beta \cap E_{f(\beta)}$ is finite". ■

Sketch of the proof

- $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ MAD, $\langle E_n : n < \omega \rangle$ $\subseteq$-decreasing. Suppose, towards a contradiction, that $f : \kappa \to \omega$ is such that $A_\alpha \cap E_{f(\alpha)}$ is finite for every α.
- Using the hypothesis and maximality, we inductively construct a sequence $\langle \alpha_n : n < \omega \rangle$ of distinct ordinals in κ and a increasing sequence of naturals $\langle k_n : n < \omega \rangle$ such that, for every $n < \omega$, $A_{\alpha_n} \cap E_{k_n}$ is infinite but $A_{\alpha_n} \cap E_{k_{n+1}}$ is finite ("take $k_{n+1} = f(\alpha_n)$", etc.)
- Define the disjoint family $\{B_n : n < \omega\}$, $B_n = A_{\alpha_n} \setminus \bigcup_{m<n} A_{\alpha_m}$. Notice that each $B_n \cap E_{k_n}$ is infinite (and, therefore, non-empty !).
- Pick $x_n \in B_n \cap E_{k_n}$ and let C be the infinite set $C = \{x_n : n < \omega\}$.
- By maximality, there is some β such that $C \cap A_\beta$ is an infinite set, let $C \cap A_\beta = \{x_{n_i} : i < \omega\}$.
- Notice that if $n_i > n$ then $x_{n_i} \in E_{k_n}$ (the sequence of E_n's is decreasing !), and therefore $A_\beta \cap E_{k_n}$ is infinite for every $n < \omega$.
- As k_n's increase and E_n's decrease, it follows that $A_\beta \cap E_n$ is infinite for every $n < \omega$ – and this contradicts "$A_\beta \cap E_{f(\beta)}$ is finite". ∎

Sketch of the proof

- $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ MAD, $\langle E_n : n < \omega \rangle$ $\subseteq$-decreasing. Suppose, towards a contradiction, that $f : \kappa \to \omega$ is such that $A_\alpha \cap E_{f(\alpha)}$ is finite for every α.
- Using the hypothesis and maximality, we inductively construct a sequence $\langle \alpha_n : n < \omega \rangle$ of distinct ordinals in κ and a increasing sequence of naturals $\langle k_n : n < \omega \rangle$ such that, for every $n < \omega$, $A_{\alpha_n} \cap E_{k_n}$ is infinite but $A_{\alpha_n} \cap E_{k_{n+1}}$ is finite ("take $k_{n+1} = f(\alpha_n)$", etc.)
- Define the disjoint family $\{B_n : n < \omega\}$, $B_n = A_{\alpha_n} \setminus \bigcup_{m<n} A_{\alpha_m}$. Notice that each $B_n \cap E_{k_n}$ is infinite (and, therefore, non-empty !).
- Pick $x_n \in B_n \cap E_{k_n}$ and let C be the infinite set $C = \{x_n : n < \omega\}$.
- By maximality, there is some β such that $C \cap A_\beta$ is an infinite set, let $C \cap A_\beta = \{x_{n_i} : i < \omega\}$.
- Notice that if $n_i > n$ then $x_{n_i} \in E_{k_n}$ (the sequence of E_n's is decreasing !), and therefore $A_\beta \cap E_{k_n}$ is infinite for every $n < \omega$.
- As k_n's increase and E_n's decrease, it follows that $A_\beta \cap E_n$ is infinite for every $n < \omega$ – and this contradicts "$A_\beta \cap E_{f(\beta)}$ is finite". ∎

Sketch of the proof

- $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ MAD, $\langle E_n : n < \omega \rangle$ $\subseteq$-decreasing. Suppose, towards a contradiction, that $f : \kappa \to \omega$ is such that $A_\alpha \cap E_{f(\alpha)}$ is finite for every α.
- Using the hypothesis and maximality, we inductively construct a sequence $\langle \alpha_n : n < \omega \rangle$ of distinct ordinals in κ and a increasing sequence of naturals $\langle k_n : n < \omega \rangle$ such that, for every $n < \omega$, $A_{\alpha_n} \cap E_{k_n}$ is infinite but $A_{\alpha_n} \cap E_{k_{n+1}}$ is finite ("take $k_{n+1} = f(\alpha_n)$", etc.)
- Define the disjoint family $\{B_n : n < \omega\}$, $B_n = A_{\alpha_n} \setminus \bigcup_{m<n} A_{\alpha_m}$. Notice that each $B_n \cap E_{k_n}$ is infinite (and, therefore, non-empty !).
- Pick $x_n \in B_n \cap E_{k_n}$ and let C be the infinite set $C = \{x_n : n < \omega\}$.
- By maximality, there is some β such that $C \cap A_\beta$ is an infinite set, let $C \cap A_\beta = \{x_{n_i} : i < \omega\}$.
- Notice that if $n_i > n$ then $x_{n_i} \in E_{k_n}$ (the sequence of E_n's is decreasing !), and therefore $A_\beta \cap E_{k_n}$ is infinite for every $n < \omega$.
- As k_n's increase and E_n's decrease, it follows that $A_\beta \cap E_n$ is infinite for every $n < \omega$ – and this contradicts "$A_\beta \cap E_{f(\beta)}$ is finite". ■

Versions of countable paracompactness

For now on, X is a topological space and κ is a regular cardinal.

A few definitions

- An open family $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ is a **shrinking** of an open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of X if $\mathcal{V}$ is also an open cover of X and for all $\alpha < \kappa$ we have $\overline{V_\alpha} \subseteq U_\alpha$.
- An open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of X is **monotone** if $\forall \alpha < \beta < \kappa$ we have $U_\alpha \subsetneq U_\beta$.

Monotone open covers also appear under various other names in the literature, *e.g.* **ascending open covers**, or – and the following terminology was widely used in the 70's and 80's – **nested open covers**.

Versions of countable paracompactness

For now on, X is a topological space and κ is a regular cardinal.

A few definitions

- An open family $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ is a **shrinking** of an open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of X if $\mathcal{V}$ is also an open cover of X and for all $\alpha < \kappa$ we have $\overline{V_\alpha} \subseteq U_\alpha$.
- An open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of X is **monotone** if $\forall \alpha < \beta < \kappa$ we have $U_\alpha \subsetneq U_\beta$.

Monotone open covers also appear under various other names in the literature, *e.g.* **ascending open covers**, or – and the following terminology was widely used in the 70's and 80's – **nested open covers**.

A list of properties, inspired by M. E. Rudin

"...the dreadful names are unfortunately historical ...",
Mary Ellen Rudin, 1985

Recall that κ is always supposed to be a regular cardinal ...

(i) X is κ-**paracompact** if every open cover $\mathcal{U}$ of X of size κ and with $X \notin \mathcal{U}$ has a locally finite refinement.

(ii) X is κ-**B** if every monotone open cover of X of order type κ has a monotone shrinking.

(iii) X is κ-**shrinking** if every open cover of size κ with $X \notin \mathcal{U}$ has a shrinking.

(iv) X is κ-**D** if every monotone open cover of X of order type κ has a shrinking.

If $\kappa = \aleph_0$, all of the above are equivalent ! ... Remark: these are not precisely Rudin's definitions ...

Our versions are, indeed, versions – not strengthenings

Our definitions here are different from those made by M. E. Rudin in 1985.

What Mary Ellen Rudin defined as κ-paracompact, etc., is what we called in our paper as $\leqslant \kappa$-paracompact, etc. – that is, Rudin's definitions were made with the presented requirements done for every regular cardinal $\lambda \leqslant \kappa$, and so, in her case, properties were indeed **strengthenings** of countable paracompactness.

However, or versions conform with the usage of later authors: for instance, our notions of κ-B and κ-D are precisely, respectively, the $\mathcal{B}(\kappa)$-property and the $\mathcal{B}^*(\kappa)$-property as defined by Yasui in 1983.

The choice of the letter "B" is related to the $\mathcal{B}$*-property*, introduced by Zenor in 1970 as a strengthening of countable paracompactness (a space is said to satisfy the $\mathcal{B}$-property if every monotone open cover has a monotone shrinking).

Relatively κ-D and relatively κ-B subsets

Given the κ-versions listed, and also considering the presented equivalences of relative countable paracompactness, we came up with the following new notions:

Two new relative topological properties (M., R., da S. – 2015)

Let X be a topological space and let Y be a subset of X.

We say that Y is **relatively κ-D** (resp. **relatively κ-B**) **in** X if for every decreasing sequence $\langle C_\alpha : \alpha < \kappa \rangle$ of closed subsets of X such that $\bigcap_{i<\kappa} C_\alpha = \emptyset$, there is a sequence of open sets (resp. decreasing sequence of open sets) $\langle A_\alpha : \alpha < \kappa \rangle$ satisfying $C_\alpha \cap Y \subseteq A_\alpha$ for each $\alpha < \kappa$ and such that $\bigcap_{\alpha<\kappa} \overline{A_\alpha} \cap Y = \emptyset$.

The desired flavour of a relative topological property

Let X be a topological space, Y be a subset of X and $\mathcal{U} = \{U_i : i \in I\}$ be an indexed open cover of X.

We will say that $\mathcal{U}$ **has a relative shrinking with respect to** Y if there is a family of open sets $\mathcal{V} = \{V_i : i \in I\}$ which is the relative shrinking of $\mathcal{U}$, meaning that $\overline{V_i} \cap Y \subseteq U_i$ for every $i \in I$ and $Y \subseteq \bigcup \mathcal{V}$.

The following proposition (whose proof we omit in this talk) brings us the desired flavour of a relative topological property for our definition of relative κ-D.

Proposition (Morgan, Rangel, da S. – 2015)

Let X be a topological space and $Y \subseteq X$. The following statements are equivalent, for every regular κ:

(i) Y is relatively κ-D in X.
(ii) Every monotone open cover of X of order type κ has a relative shrinking with respect to Y.

κ-D and κ-B almost disjoint families

Suppose κ is a regular cardinal and $\mathcal{P}$ is one of κ-paracompact, κ-B, κ-D, κ-shrinking. An almost disjoint family $\mathcal{A}$ will be said **to satisfy** $\mathcal{P}$ (or, simply, $\mathcal{A}$ **is** $\mathcal{P}$) if $\mathcal{A}$ is **relatively** $\mathcal{P}$ in its corresponding $\Psi(\mathcal{A})$.

Characterization of κ-D almost disjoint families (M., R., da S. – 2015)

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an a.d. family and consider the corresponding space $\Psi(\mathcal{A})$. The following statements are equivalent:

(*i*) $\mathcal{A}$ is κ-D.
(*ii*) For every decreasing sequence $\langle \mathcal{F}_\alpha : \alpha < \kappa \rangle$ of subsets of $\mathcal{A}$ such that $\bigcap_{\alpha<\kappa} \mathcal{F}_\alpha = \emptyset$ there is a sequence $\langle E_\alpha : \alpha < \kappa \rangle$ of subsets of ω such that:

(*ii*).1 $\forall \alpha < \kappa \ \forall A \in \mathcal{F}_\alpha \ (A \setminus E_\alpha \text{ is finite})$; and
(*ii*).2 $\forall A \in \mathcal{A} \ \exists \alpha < \kappa \ (A \cap E_\alpha \text{ is finite})$.

Some results on κ-B and κ-D

Of course, in order to characterize κ-B, one should look at strictly decreasing sequences $\langle E_\alpha : \alpha < \kappa \rangle$ – but, of course, there are some clear restrictions on this ! So, we have

Proposition (Morgan, Rangel, da S. – 2015)

If κ is an uncountable regular cardinal and $\mathcal{A}$ is an a.d. family with $|\mathcal{A}| \geqslant \kappa$ then $\mathcal{A}$ is not κ-B.

However, for κ-D we have the following:

Theorem (Morgan, Rangel, da S. – 2015)

If $\mathcal{A}$ is an a.d. family of size κ, then it is κ-D.

We gave a combinatorial proof for the preceding theorem – however, one could argue topologically and check that

every regular space of size κ is κ-D – recall that κ is always assumed to be regular !!!

Weakly κ-B subsets

Given the strong restrictions on strictly $\subseteq$-decreasing sequences of subsets of ω, it is natural to consider sequences which are decreasing in the sense of $\subseteq^*$ ("almost inclusion").

This attempt is also justified by the following fact, whose verification we omit in this talk:

Let $\mathcal{A}$ be an a.d. family. The following are equivalent.

(i) $\Psi(\mathcal{A})$ is countably paracompact.

(ii) For every function $g : \mathcal{A} \to \omega$ there is a $\subseteq^*$-**decreasing** sequence $\langle E_n : n < \omega \rangle$ of subsets of ω and a function $f : \mathcal{A} \to \omega$ such that

(ii).1 $\forall A \in \mathcal{A}$ $(A \setminus E_{g(A)}$ is finite); and
(ii).2 $\forall A \in \mathcal{A}$ $(A \cap E_{f(A)}$ is finite).

(That is, for $\kappa = \aleph_0$ we were already allowed to consider $\subseteq^*$ instead of $\subseteq$ . . .)

Weakly κ-B subsets

To work with $\subseteq^*$-decreasing sequences

We say that an a.d. family $\mathcal{A}$ is **weakly κ-B** (or **relatively weakly κ-B** in $\Psi(\mathcal{A})$) if for every strictly decreasing sequence $\langle \mathcal{F}_\alpha : \alpha < \kappa \rangle$ of subsets of $\mathcal{A}$ such that $\bigcap_{\alpha<\kappa} \mathcal{F}_\alpha = \emptyset$ there is a $\subseteq^*$-decreasing sequence $\langle E_\alpha : \alpha < \kappa \rangle$ of subsets of ω satisfying the conditions:

(i) $\forall \alpha < \kappa \; \forall A \in \mathcal{F}_\alpha \; (A \setminus E_\alpha \text{ is finite})$; and
(ii) $\forall A \in \mathcal{A} \; \exists \alpha < \kappa \; (A \cap E_\alpha \text{ is finite})$.

Clearly,

$$\mathcal{A} \text{ is } \kappa\text{-B} \Longrightarrow \mathcal{A} \text{ is weakly } \kappa\text{-B} \Longrightarrow \mathcal{A} \text{ is } \kappa\text{-D}.$$

Results on weakly κ-B subsets

We have proved:

If there is an a.d. family of size θ which is weakly κ-B, then there is a dominating family of size $\leqslant 2^\kappa$ in ${}^\theta\kappa$.

If $\theta = \aleph_1$ and $\kappa = \aleph_0$, the existence of such **small dominating family** is related to large cardinals, as already remarked.

However, our main result on weakly κ-B a.d. families is the following one:

Theorem (Morgan, Rangel, da S. – 2015)

If $\mathcal{A}$ is a MAD family and $\kappa < \mathfrak{t}$, then $\mathcal{A}$ is not weakly κ-B.

Notice that, in particular, the preceding theorem is a strengthening of the one asserting that "MAD families are not countably paracompact".

Proof of: "A MAD family $\mathcal{A}$ is not weakly κ-B for $\kappa < \mathfrak{t}$"

It suffices to prove that: if $\kappa < \mathfrak{t}$, $\mathcal{A}$ is a MAD family and $\langle E_\alpha : \alpha < \kappa \rangle$ is a $\subseteq^*$-decreasing sequence of infinite subsets of ω, then there is no $f : \mathcal{A} \to \kappa$ such that $A \cap E_{f(A)}$ is a finite set for every $A \in \mathcal{A}$.

Suppose towards a contradiction that $f : \mathcal{A} \to \kappa$ is a function satisfying $|A \cap E_{f(A)}| < \omega$ for every $A \in \mathcal{A}$. As $\kappa < \mathfrak{t}$, we may consider an infinite pseudo-intersection of the decreasing sequence, say E. By maximality of $\mathcal{A}$, there is $A \in \mathcal{A}$ such that $A \cap E$ is an infinite set. However, one has

$$A \cap E \subseteq^* A \cap E_{f(A)},$$

and this is clearly an absurd. ■

κ^+-Luzin gaps are not κ-D

Let us consider the following **generalization of Luzin gaps**:

κ-Luzin gaps (Fuchino, Soukup 1997)

Let κ be a regular cardinal. An a.d. family $\mathcal{A}$ of size κ is said to be a κ-**Luzin gap** if no two disjoint subfamilies of size κ can be separated, *i.e.*, if $\mathcal{B}$, $\mathcal{C} \subseteq \mathcal{A}$ and $|\mathcal{B}| = |\mathcal{C}| = \kappa$ then there is no $E \subseteq \omega$ such that $A \cap E$ is finite for all $A \in \mathcal{B}$ and $A \subseteq^* E$ for all $A \in \mathcal{C}$.

The following theorem generalizes the main result of Morgan, Hrušák, da Silva 2012:

Theorem (Morgan, Rangel, da S. – 2015)

If $|\mathcal{A}| = \kappa^+$ and $\mathcal{A}$ is κ-D, then it is not a κ^+-Luzin gap.

Notes, questions and problems

Fuchino and Soukup (1997) have investigated the κ-**Freese-Nation property**, a property related to lattices. In the final section of their paper they use the concept to prove a result about almost disjoint families and Luzin gaps. We give a formulation of the notion for almost disjoint families.

The κ-FN property for almost disjoint families

Let κ be a regular cardinal, $\kappa \leqslant \mathfrak{c}$. If $\mathcal{A} \subseteq [\omega]^\omega$ is an almost disjoint family then $f : \mathcal{A} \longrightarrow [\mathcal{P}(\omega)]^{<\kappa}$ is a κ-**FN function for** $\mathcal{A}$ if for all distinct $a, b \in \mathcal{A}$ there is some $c \in f(a) \cap f(b)$ such that $a \subseteq^* c \subseteq^* \omega \setminus b$. The family $\mathcal{A}$ has the κ-**FN property** if there is some κ-FN function for $\mathcal{A}$.

κ-Freese-Nation Property

The following follows from addapting Fuchino/Soukup proof for $\kappa = \omega_1$.

κ-FN avoids κ^+-Luzin for κ^+-sized a.d. families

Let κ be a regular cardinal. If $\mathcal{A}$ is an a.d. family of size κ^+ and has the κ-FN property then $\mathcal{A}$ is not a κ^+-Luzin gap.

So, we have identified some similarity between κ-D and κ-FN: both of them avoid the presence of the κ^+-Luzin property in κ^+-sized a. d. families. So, we ask:

Question

Is there any relationship between an a.d. family having the κ-FN property and being κ-D ? In particular, if $\mathcal{A}$ has the κ-FN property is it necessarily κ-D ?

Analogues ? New characterizations ?

Notice that we effectively dealt in this paper only with κ-D, κ-B and weakly κ-B almost disjoint families.

Problem

Determine which results of this paper have valid analogous for κ-paracompact and κ-shrinking a.d. families. Determine combinatorial characterizations, if any, of a.d. families satisfying any of the relative κ-paracompactness type properties presented.

We are also interested in the following:

Problem

Characterize combinatorially, if possible, the almost disjoint families $\mathcal{A}$ which satisfy, if any, the following property: for every open cover of $\Psi(\mathcal{A})$ with order type κ there is a family of open sets which refines the open cover and is locally smaller than κ at $\mathcal{A}$.

The above condition easily implies κ-D, but the authors don't believe that these two notions are equivalent (for uncountable regular values of κ).

Around κ-D and MAD families

We have proved that MAD families are not countably paracompact – so, they are none of $\aleph_0$-D, $\aleph_0$-B or weakly $\aleph_0$-B (since, as remarked *en passant*, all of these properties are equivalent to countable paracompactness when $\kappa = \aleph_0$).

We have also proved that MAD families are not weakly κ-B for $\kappa < \mathfrak{t}$.

Considering these results, we can ask a number of questions:

Around κ-D and MAD families

Is it true that if $\mathcal{A}$ is κ-D (for some $\kappa > \aleph_0$), then $\mathcal{A}$ is not MAD ?

More strongly, suppose that $\mathcal{A}$ is an a.d. family, $\kappa < |\mathcal{A}|$ and there is a sequence of sets $\langle E_\alpha : \alpha < \kappa \rangle$ s. t. $\forall A \in \mathcal{A}\ \exists \alpha < \kappa\ (A \cap E_\alpha \text{ is finite})$. The following questions are posed for $\mathcal{A}$ under these assumptions.

Is $\mathcal{A}$ is necessarily not MAD ?
Is $\mathcal{A}$ is necessarily not κ^+-Luzin ?
What about the latter question if we strengthen the hypothesis to $\langle E_\alpha : \alpha < \kappa \rangle$ being a $\subseteq^*$-decreasing sequence (for $\kappa < \mathfrak{t}$)?

Notice that the proof we give for "MAD families are not weakly κ-B for $\kappa < \mathfrak{t}$" provides a positive answer for the analogue of the former question in the previous box for $\subseteq^*$-decreasing sequences.

Of course, the answers for almost all of the questions posed in this last part of the talk may depend on specific values of κ.

References

Fuchino, S., and Soukup, L., *More set-theory around the weak Freese-Nation property*, European Summer Meeting of the Association for Symbolic Logic (Haifa, 1995), Fundamenta Mathematicae **154**, 2 (1997), 159–176.

Hrušák, M., Morgan, C. J. G., and da Silva, S. G., *Luzin gaps are not countably paracompact.* Questions and answers in general topology, **30** (2012), 59–66.

Morgan, C. J. G. and da Silva, S. G., *Almost disjoint families and "never" cardinal invariants*. Commentationes Mathematicae Universitatis Carolinae, **50**, 3 (2009), 433–444.

Rudin, M. E., *κ-Dowker spaces*, in: Aspects of topology, London Math. Soc. Lecture Note Ser., Volume **93**, Cambridge Univ. Press, Cambridge, 1985, 175–193.

da Silva, S.G., *On the presence of countable paracompactness, normality and property (a) in spaces from almost disjoint families*, Questions and Answers in General Topology **25**, 1 (2007), 1–18.

da Silva, S. G., *Large cardinals and topology: a short retrospective and some new results*, Logic Journal of the IGPL **15**, 5-6 (2007), 433-443.

Watson, W., *Separation in countably paracompact spaces*, Transactions of Amererican Mathematical Society **290** (1985), 831-842.

Yasui, Y., *On the characterization of the $\mathcal{B}$-property by the normality of product spaces*, Topology and its Applications **15**, 3 (1983), 323–326.

Zenor, P., *A class of countably paracompact spaces*, Proceedings of the American Mathematical Society **24** (1970), 258–262.

Thanks and I hope see you soon in Salvador !