Lindelöf spaces and large cardinals

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Abstract:
We are going to show some connections between large cardinals and Lindelöf spaces with small pseudocharacter. Especially, we see that some large cardinals would be needed to settle an Arhangel'skii's question about cardinality of Lindelöf space with points G_δ.

Arhangel'skii's inequality

All topological spaces are assumed to be T_1.

A space X is Lindelöf if every open cover has a countable subcover.

Theorem 1 (Arhangel'skii (1969))

If X is Hausdorff, Lindelöf, and first countable, then the cardinality of X is $\leq 2^{\aleph_0}$.

Theorem 2 (Arhangel'skii)

If X is Hausdorff, then $|X| \leq 2^{L(X)+\chi(X)}$.

- $L(X)$, Lindelöf number of X, is the least infinite cardinal κ such that every open cover of X has a subcover of size $\leq \kappa$.
- $\chi(X)$: the character of X.

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Revised Arhangel'skii's inequality

Theorem 3 (Arhangel'skii, Shapirovskii)

If X is Hausdorff, then $|X| \leq 2^{L(X)+t(X)+\psi(X)}$.

Definition 4

- For $x \in X$, $\psi(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a family of open sets, $\bigcap \mathcal{U} = \{x\}\} + \aleph_0$.
- The pseudocharacter of X, $\psi(X)$, is $\sup\{\psi(x, X) : x \in X\}$.

- $t(X)$, the titghtness number of X, is the least infinite cardinal κ such that for every $A \subseteq X$ and $x \in \overline{A}$, there is $B \subseteq A$ of size $\leq \kappa$ such that $|B| \leq \kappa$ and $x \in \overline{B}$.

Note that $\psi(X) + t(X) \leq \chi(x)$.

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Note that $\psi(X) + t(X) \leq \chi(x)$.

Arhangel'skii's question

Remark 5

1. There is an arbitrary large compact Hausdorff space with countable tightness.
2. There is an arbitrary large space with countable tightness and points G_δ. A space X is with points G_δ if for each $x \in X$, the set $\{x\}$ is a G_δ-set $\iff \psi(X) = \aleph_0$.

Question 6 (Arhangel'skii (1969))

Suppose X is Hausdorff, Lindelöf, and with points G_δ, does $|X| \leq 2^{\aleph_0}$?
In other words, does $|X| \leq 2^{L(X)+\psi(X)}$?

This question is not settled completely, but we have some partial answers.

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Partial answers: Forcing constructions

By forcing methods, Shelah showed the consistency of the existence of a large Lindelöf space with points G_δ.

Theorem 7 (Shelah (1978), Gorelic (1993))

It is consistent that ZFC+Continuum Hypothesis+"there exists a regular Lindelöf space with points G_δ and of size $2^{\aleph_1}$ (+ $2^{\aleph_1}$ is arbitrary large)".

Partial answers: Construction using $\diamondsuit^*$

Theorem 8 (Dow (2015))

Suppose $\diamondsuit^*$ holds, that is, there exists $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ such that

1. $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{A}_\alpha| \leq \omega$.
2. For every $A \subseteq \omega_1$, the set $\{\alpha < \omega_1 : A \cap \alpha \in \mathcal{A}_\alpha\}$ contains a club in ω_1.

Then there exists a zero-dimensional Hausdorff Lindelöf space with points G_δ and of size $2^{\aleph_1}$.

Note that $\diamondsuit^*$ (even $\diamondsuit$) implies CH, and CH is consistent with no $\diamondsuit$.

Partial answers: Spaces with topological games

Let X be a topological space, and α an ordinal. Let G_α denote the following topological game of length α:

ONE	$\mathcal{U}_0$	$\mathcal{U}_1$	$\cdots$	$\mathcal{U}_\xi$	$\cdots$	($\mathcal{U}_\xi$: open cover of X)
TWO	O_0	O_1	$\cdots$	O_ξ	$\cdots$	($O_\xi \in \mathcal{U}_\xi$: open set)

For a play $\langle \mathcal{U}_\xi, O_\xi : \xi < \alpha \rangle$, TWO wins if $\{O_\xi : \xi < \alpha\}$ is an open cover of X.

Definition 9

X satisfies G_α if the player ONE in the game G_α on X does not have a winning strategy.

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Theorem 10 (Pawlikowski)

X satisfies G_ω if, and only if, X is Rothberger.

X is Rothberger if for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of X, there is $\langle O_n : n < \omega \rangle$ such that $O_n \in \mathcal{U}_n$ and $\{O_n : n < \omega\}$ is an open cover.

Definition 11 (Scheepers-Tall (2000))

A Lindelöf space X is indestructible if X satisfies G_{ω_1}.

Theorem 12 (Scheepers-Tall)

A Lidelöf space X is indestructible if for every σ-closed forcing $\mathbb{P}$, $\mathbb{P}$ forces that "X is Lindelöf"

$\iff$ there exists a family of open sets $\langle O_s : s \in {}^{<\omega_1}\omega \rangle$ such that

1. For $s \in {}^{<\omega_1}\omega$, $\{O_{s^\frown\langle n \rangle} : n < \omega\}$ is an open cover of X.
2. There is no $f : \omega_1 \to \omega$ such that $\{O_{f\restriction\alpha} : \alpha < \omega_1\}$ covers X.

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Some examples

Theorem 13 (Scheepers-Tall)

The following are indestructibly Lindelöf spaces:

1. Second countable spaces.
2. Lindelöf spaces with size $\aleph_1$.
3. Rothberger spaces.

These spaces are (consistently) small spaces as cardinality $\leq 2^{\aleph_0}$.

A typical example of destructible large space is:

1. A product space $2^{\aleph_1}$, which is compact, weight $\aleph_1$, and size $2^{\aleph_1}$.

There may be no large indestructibly Lindelöf spaces

Theorem 14 (Tall (1995), Scheepers-Tall, Tall-Usuba (2014))

1. If κ is a measurable cardinal, then the Levy collapse $\mathrm{Col}(\omega_1, < \kappa)$ forces that "there is no indestructibly Lindelöf space with $\psi(X) \leq \aleph_1$ and of cardinality $> 2^{\aleph_0}$".
2. If κ is a weakly compact cardinal, then the Levy collapse $\mathrm{Col}(\omega_1, < \kappa)$ forces that "there is no indestructibly Lindelöf space with $\psi(X) \leq \aleph_1$ and of cardinality $\aleph_2$".

So no large indestructibly Lindelöf space with points G_δ is consistent modulo large cardinal.

There may be large indestructibly Lindelöf spaces

Theorem 15 (Dias-Tall (2014))

If ω_2 is not a weakly compact cardinal in the Gödel's constructible universe L, then there is a regular indestructibly Lindelöf, compact space with $\psi(X) \leq \aleph_1$ and of cardinality $\aleph_2$.

Corollary 16

The following are equiconsistent:

1. There exists a weakly compact cardinal.
2. There is no regular indestructibly Lindelöf space with $\psi(X) \leq \aleph_1$ and of size $\aleph_2$.

This shows that the statement "no large indestructibly Lindelöf space with points G_δ" is a large cardinal property, at least greater than the existence of a weakly compact cardinal.

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Upper bound by Large cardinals

Theorem 17

1. (Shelah) If κ is a weakly compact cardinal, then there is no space X such that $|X| = \kappa$ and $\psi(X), L(X) < \kappa$.
2. (Arhangel'skii) If κ is the least measurable cardinal, then there is no space X such that $|X| \geq \kappa$ and $\psi(X), L(X) < \kappa$.

Corollary 18

1. If κ is weakly compact, then there is no Lindelöf space with points G_δ and of size κ.
2. If κ is the least measurable cardinal, then every Lindelöf space with points G_δ has cardinality $< \kappa$.

Arhangel'skii's question

Question 19 (recall, still open)

Is it consistent that that ZFC+"every regular (or Hausdorff) Lindelöf space with points G_δ has cardinality $\leq 2^{\aleph_0}$"?

Known results indicate that this problem is connecting with large cardinals, and, even if it is consistent, we would need some large cardinal to show the consistency.

Main theorem

Inspired by Dow's construction using $\diamondsuit^*$, under some extra assumptions, we have another construction of regular Lindelöf spaces with points G_δ

Theorem 20

Suppose that either:

1. There exists a regular Lindelöf P-space with pseudocharacter $\leq \aleph_1$ and of size $> 2^{\aleph_0}$,
2. CH+there exists an ω_1-Kurepa tree, or
3. CH+$\square(\omega_2)$holds.

Then there exists a regular Lindelöf space with points G_δ and of size $> 2^{\aleph_0}$.

Theorem 21 (Todorcevic)

If $\square(\kappa)$ fails for some regular uncountable κ, then κ is weakly compact in L.

Corollary 22

If

- ZFC+CH+ "every regular Lindelöf space with points G_δ has cardinality $\leq 2^{\aleph_0}$"

is consistent, then so is

- ZFC+"there exists a weakly compact cardinal".

This means that, even if it is possible to construct a model in which "every regular Lindelöf space with points G_δ has cardinality $\leq 2^{\aleph_0}$", we must need a large cardinal to construct such a model.

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Combining Dias and Tall's argument with the theorem, we can also show the following:

Theorem 23

Suppose CH. Suppose $\square(\omega_2)$ holds. Then there is a regular indestructibly Lindelöf space with points G_δ and of size $\aleph_2$.

Corollary 24

The following are equiconsistent:

1. There exists a weakly compact cardinal.
2. There is no regular indestructibly Lindelöf space with $\psi(X) \leq \aleph_1$ and of size $\aleph_2$.
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Key lemma 1

Lemma 25

Let Y be an uncountable regular Lindelöf space such that:

1. $\psi(Y) \leq \aleph_1$.
2. For $y \in Y$, if $\psi(y, Y) = \aleph_1$ then there exists $\langle O^y_\alpha : \alpha < \omega_1 \rangle$ such that
   - 2.1 O^y_α is clopen.
   - 2.2 $O^y_\alpha \supseteq O^y_{\alpha+1}$.
   - 2.3 $O^y_\alpha = \bigcap_{\beta<\alpha} O^y_\beta$ if α is a limit ordinal.
   - 2.4 $\bigcap_{\alpha<\omega_1} O^y_\alpha = \{y\}$.

Then there exists a regular Lindelöf space with points G_δ and of size $|Y|$.

If Y is a regular Lindelöf P-space of pseudocharacter $\leq \aleph_1$, then Y satisfies the assumptions of Lemma 25.

Theorem 26

Suppose that there exists a regular Lindelöf P-space of pseudocharacter $\leq \aleph_0$ and of size $> 2^{\aleph_0}$. Then there exists a regular Lindelöf space with points G_δ and of size $> 2^{\aleph_0}$.

Remark 27

The statement that "(CH+) there exists a regular Lindelöf P-space of pseudocharacter $\leq \aleph_1$ and of size $> 2^{\aleph_0}$" is independent from ZFC.

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Sketch of Proof

Fix a topology $\mathcal{T}$ on ω_1 such that:

- $\langle \omega_1, \mathcal{T} \rangle$ is regular Lindelöf, and with point G_δ.

(To simplify our argument, we assume $\psi(y, Y) = \aleph_1$ for every $y \in Y$).
The underlying set of our space X is $Y \times \omega_1$.
For $A \subseteq Y$, put $[A] = A \times \omega_1$.
For $y \in Y$, $\alpha < \omega_1$, and $V \subseteq \omega_1$ open in ω_1, let

$$O(y, \alpha, V) = \bigcup \{[O^y_\beta] \setminus [O^y_{\beta+1}] : \alpha \leq \beta \in V\} \cup (\{y\} \times V).$$

(so $[O^y_\alpha] = O(y, \alpha, \omega_1)$).
$O(y, \alpha, V)$ will form a local base for $\langle y, \xi \rangle$ for $\xi \in V$.

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Our space X is $Y \times \omega_1$ equipped with the topology generated by the $O(y, \alpha, V)$'s.

Remark 28

Our topology on the space X is stronger than on the product space $Y \times \omega_1$.

Claim

X is with points G_δ.

For $y \in Y$ and $\xi \in \omega_1$, fix open V_n in ω_1 $(n < \omega)$ such that $\bigcap_n V_n = \{\xi\}$. Then $\bigcap_n O(y, \xi + 1, V_n) = \{\langle y, \xi \rangle\}$.

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For a family $\mathcal{U}$ of open sets in X, if $\{y\} \times \omega_1 \subseteq \bigcup \mathcal{U}$ then there is a countable $\mathcal{U}' \subseteq \mathcal{U}$ and $\alpha < \omega_1$ such that $[O^y_\alpha] \subseteq \bigcup \mathcal{U}'$.

Claim

X is Lindelöf.

Suppose $\mathcal{U}$ is an open cover of X. For $y \in Y$, there is $\alpha_y < \omega_1$ and a countable $\mathcal{U}_y \subseteq \mathcal{U}$ such that $[O^y_{\alpha_y}] \subseteq \bigcup \mathcal{U}_y$.

Since Y is Lindelöf and $\{O^y_{\alpha_y} : y \in Y\}$ is an open cover of Y, there is $y(n) \in Y$ $(n < \omega)$ such that $Y \subseteq O^{y(0)}_{\alpha_{y(0)}} \cup O^{y(1)}_{\alpha_{y(1)}} \cup \cdots$.

Then $\mathcal{U}_{y(0)} \cup \mathcal{U}_{y(1)} \cup \cdots$ is a countable subcover of $\mathcal{U}$.

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Then $\mathcal{U}_{y(0)} \cup \mathcal{U}_{y(1)} \cup \cdots$ is a countable subcover of $\mathcal{U}$.

Some variations

Theorem 29

Suppose CH. For each uncountable cardinal κ, there is a poset which is σ-closed, $\aleph_2$-c.c., and forces that "there exists a regular Lindelöf space with points G_δ and of size just κ (and $2^{\aleph_1} \geq \kappa$)".

So, e.g., $CH + \exists$ regular Lindelöf space with points G_δ and of size $\aleph_\omega$ is consistent.

Theorem 30

It is consistent that GCH+for each regular cardinal κ, there is a regular space X such that $|X| = 2^{2^\kappa}$, with points G_δ, and $L(X) \leq \kappa$. So $|X| \not\leq 2^{L(X)+\psi(X)}$ everywhere. Moreover this statement is consistent with almost all large cardinals.

Some variations

Theorem 29

Suppose CH. For each uncountable cardinal κ, there is a poset which is σ-closed, $\aleph_2$-c.c., and forces that "there exists a regular Lindelöf space with points G_δ and of size just κ (and $2^{\aleph_1} \geq \kappa$)".

So, e.g., $CH + \exists$ regular Lindelöf space with points G_δ and of size $\aleph_\omega$ is consistent.

Theorem 30

It is consistent that GCH+for each regular cardinal κ, there is a regular space X such that $|X| = 2^{2^\kappa}$, with points G_δ, and $L(X) \leq \kappa$. So $|X| \not\leq 2^{L(X)+\psi(X)}$ everywhere. Moreover this statement is consistent with almost all large cardinals.

Key lemma 2

$T \subseteq {}^{<\omega_2}2$: tree

A branch of T is a maximal chain of T.

Lemma 31

Suppose that there exists a tree $T \subseteq {}^{<\omega_2}2$ such that:

1. T has no branch of size $\aleph_2$.
2. T does not contain an isomorphic copy of Cantor tree ${}^{\leq\omega}2$.

Suppose T has κ cofinal branches. Then there exists a zero-dimensional Hausdorff indestructibly Lindelöf space Y with points G_δ and of size $\max\{|T|, \kappa\}$. Actually we can construct a space satisfying the assumption of Key lemma 1.

Remark 32

Suppose CH. If T is an ω_1-Kurepa tree, then

1. T has more than $2^{\aleph_0}$ many branches,
2. T does not have a branch of size $\aleph_2$, and
3. T does not contain an isomorphic copy of Cantor tree.

Theorem 33 (Todorcevic)

Suppose $\square(\omega_2)$ holds. Then there exists a tree $T \subseteq {}^{<\omega_2}2$ such that

1. T is an ω_2-Aronszajn tree.
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Corollary 34

Suppose CH+"ω_1-Kurepa tree exists", or CH+$\square(\omega_2)$ holds, then there is a regular indestructibly Lindelöf space with points G_δ and of size $> 2^{\aleph_0}$.

Theorem 35

1. The statement that "there is an ω_1-Kurepa tree" is independent from ZFC+CH (Silver).
2. The statement that "$\square(\omega_2)$ holds" is independendet from ZFC+CH (Todorcevic).

Corollary 34

Suppose CH+"ω_1-Kurepa tree exists", or CH+$\Box(\omega_2)$ holds, then there is a regular indestructibly Lindelöf space with points G_δ and of size $> 2^{\aleph_0}$.

Theorem 35

1. The statement that "there is an ω_1-Kurepa tree" is independent from ZFC+CH (Silver).
2. The statement that "$\Box(\omega_2)$ holds" is independet from ZFC+CH (Todorcevic).

Sketch of proof

(We consider only $T \subseteq {}^{<\omega_1}2$ is an ω_1-Kurepa tree)
Let $\mathcal{B}$ be the set of all branches of T.
The underlying set of our space Y is $\mathcal{B} \cup \bigcup\{T_{\alpha+1} : \alpha < \omega_1\}$. Topologize Y as follows:

1. For $t \in T_{\alpha+1}$, $\{t\}$ is open.
2. For $\alpha \leq \omega_1$ and $b : \alpha \to 2$, if b is a branch of T, then an open neighborhood of b is $\{b|\beta + 1 : \gamma \leq \beta < \alpha\}$ for some $\gamma < \alpha$.

- If $\mathrm{cf}(\mathrm{dom}(b)) = \omega$, then $\psi(b, Y) = \aleph_0$.
- If $\mathrm{cf}(\mathrm{dom}(b)) = \omega_1$, then $\psi(b, Y) = \aleph_1$ but $\{\{b|\beta + 1 : \gamma \leq \beta < \mathrm{dom}(b)\} : \gamma < \mathrm{dom}(b)\}$ is a clopen local base at $\{b\}$.
- Y is Lindelöf, because T does not contains a copy of Cantor tree.

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Simple forcing creates Lindelöf space with points G_δ

Corollary 36

Let $\mathbb{P}$ be the Cohen forcing. Then $\mathbb{P}$ forces that "there exists a regular Lindelöf space with points G_δ and of size $(2^{\aleph_1})^V$".

In $V^{\mathbb{P}}$, the tree $({}^{<\omega_1}2)^V$ does not have a copy of Cantor tree.

Question 37

1. Is it consistent that ZFC+"every regular Lindelöf space with points G_δ has cardinality $\leq 2^{\aleph_0}$"?
2. Is it consistent that ZFC+"there is a regular Lindelöf space with points G_δ and of size $> 2^{\aleph_0}$"+"Large cardinal propertis on ω_1 and ω_2 (such as stationary reflection principles)"?
3. Is it consistent that ZFC+"every regular Lindelöf c.c.c. space with points G_δ has cardinality $\leq 2^{\aleph_0}$"?

   (Gorelic's space satisfies the c.c.c.)
4. Is it consistent that ZFC+"there is a regular Lindelöf space with points G_δ and of size $> 2^{2^{\aleph_0}}$"? Or, is $2^{2^{\aleph_0}}$ a "real" upper bound of cardinalities of Lindelöf spaces with points G_δ?

Thank you for your attention!

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