MORE ON THE PROPERTIES OF ALMOST CONNECTED PRO-LIE GROUPS

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(i) every neighborhood of the identity in G contains a normal subgroup N such that G/N is a Lie group;

(ii) G is complete.

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(iii) *if N is a closed normal subgroup of a pro-Lie group G, then the quotient group G/N is a pro-Lie group provided that either N locally compact, or N is Polish, or N is almost connected and G/N is complete.*

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Every infinite-dimensional Banach space B is not a pro-Lie group.

Indeed, let $U = \{x \in B : ||x|| < 1\}$, where $|| \cdot ||$ is the norm on B. The unit ball U does not contain non-trivial subgroups, while B has infinite dimension.

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In the sequel we focus on almost connected pro-Lie groups.

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Clearly every compact group is ω-narrow. So it suffices to verify that every connected pro-Lie group is ω-narrow. The latter follows from the fact that every connected locally compact group is σ-compact and, hence, ω-narrow.

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It turns out that the answer to all of (a)–(e) is "Yes".

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Let us see some details.

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Definition 2.6.

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Fact 2.7.

Every $\mathbb{R}$-factorizable group is ω-narrow. The converse is false.

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Every separable topological group is ω-narrow, but there exists a separable topological group which fails to be $\mathbb{R}$-factorizable (Reznichenko and Sipacheva).

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There exists an ω-narrow pro-discrete (hence pro-Lie) abelian group G which fails to be $\mathbb{R}$-factorizable.

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There exists an ω-narrow pro-discrete (hence pro-Lie) abelian group G which fails to be $\mathbb{R}$-factorizable.

In fact, G is a closed subgroup of $\mathbb{Q}^{\omega_1}$, where the latter group is endowed with the ω-box topology (and the group $\mathbb{Q}$ of rationals is discrete). The projections of G to countable subproducts are countable, which guarantees that G is ω-narrow.

Main results

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Let a topological group H be a continuous homomorphic image of an almost connected pro-Lie group G. Then the following hold:

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(a) *the group H is $\mathbb{R}$-factorizable;*

(b) *the space H is ω-cellular;*

(c) *The Hewitt–Nachbin completion of H, υH, is again an $\mathbb{R}$-factorizable and ω-cellular topological group containing H as a (dense) topological subgroup.*

Some proofs

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Theorem 2.10 ("CONTINUOUS IMAGES"–Tk., 2015).

Let $X = \prod_{i \in I} X_i$ be a product space, where each X_i is a regular Lindelöf Σ-space and $f : X \to G$ a continuous mapping of X onto a regular paratopological group G.

Some proofs

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2) Clearly C and $\mathbb{R}$ are Lindelöf Σ-spaces, so H is a continuous image of a product of Lindelöf Σ-spaces. Evidently H is regular. By the Continuous Images theorem, the groups G and υG are $\mathbb{R}$-factorizable and ω-cellular.

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3) Since the dense subgroup G of the paratopological group υG is a topological group, so is υG (a result due to Iván Sánchez). $\square$

Main results

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Theorem 2.11 (Leiderman–Tk., 2015).

Let G be a topological group and K a compact invariant subgroup of G such that the quotient group G/K is homeomorphic to the product $C \times \prod_{i\in I} H_i$, where C is a compact group and each H_i is a topological group with a countable network. Then:

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In other words, every extension of a topological group H homeomorphic with $C \times \prod_{i \in I} H_i$ by a compact group has the above properties (a)–(c). Hence an extension of an almost connected pro-Lie group by a compact group has properties (a)–(c).

Main results

Problem 2.12.
Let G be a Hausdorff topological group and K a compact invariant subgroup of G such that G/K is an almost connected pro-Lie group. Is G a pro-Lie group?

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Let G be a Hausdorff topological group and K a compact invariant subgroup of G such that G/K is an almost connected pro-Lie group. Is G a pro-Lie group?

Under an additional assumption, we give the affirmative answer to the problem.

Theorem 2.13 (Leiderman-Tk., 2015).

Let G be a pro-Lie group and K a compact invariant subgroup of G such that the quotient group G/K is an almost connected pro-Lie group. Then G is almost connected.

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(2) **continuous cross-sections**.

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Let K be a compact invariant subgroup of a topological group X and $p\colon X \to X/K$ the quotient homomorphism.

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We apply the above theorem with Y being a convergent sequence (with its limit).

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Item (a) follows from Theorem 2.9, while the proof of (b) is non-trivial and requires some techniques presented in our joint work with A. Leiderman:

Lattices of homomorphisms and pro-Lie groups, arXiv:1605.05279.

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LAST MINUTE NOTE: The answer to (a) and (b) of Problem 3.1 is 'NO' [Taras Banakh].