Topological approch to "nontopological" ultrafilters.

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I-**ultrafilters** (J. E. Baumgartner 1995)
Let I be a familly of "small" subsets of some set X, i.e. I is closed for taking subsets and all finite subsets of X belong to I. We say that an ultrafilter u (on ω) is an I-ultrafilter if for each function $f : \omega \to X$ there exists a set $U \in u$ such that $f[U] \in I$.

Ordinal ultrafilters (J. E. Baumgartner 1995) If $X = \omega_1$ and J_α is a family of sets of order type less than α then we obtain J_α-ultrafilters. **Proper J_α-ultrafilters** are such J_α-ultrafilters which are not J_β-ultrafilters for any $\beta < \alpha$. The class of proper J_α ultrafilters we denote by J^*_α.

Theorem (Baumgartner)

If an ultrafilter u is J^*_α-ultrafilter then α is indecomposable, i.e. $\alpha = \omega^\beta$ for some ordinal number β.

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Level ultrafilters (J. E. Baumgartner 1995) Now let $X = \mathbb{R}$ with a standard topology, and consider families L_α of sets which derivative of rank α is empty. Proper L_α-ultrafilters are such L_α-ultrafilters that are not L_β-ultrafilters for any $\beta < \alpha$. This way we obtain a hierarchy of level ultrafilters.

Let (X, τ) be any topological space, we consider families $L_\alpha^{(X,\tau)}$ of sets which derivative of rank α (in (X, τ)) is empty. Proper $L_\alpha^{(X,\tau)}$-ultrafilters are such $L_\alpha^{(X,\tau)}$-ultrafilters that are not $L_\beta^{(X,\tau)}$-ultrafilters for any $\beta < \alpha$. This way we obtain a hierarchy of level (X, τ) ultrafilters.

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Proper J_{ω^α}-ultrafilters are precisely proper $L_\alpha^{(\omega_1,\mathrm{ord})}$-ultrafilters.

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If $(u_n)_{n<\omega}$ is a sequence of filters on ω and v is a filter on ω then the **Frolick sum = limite = contour on the sequence** (u_n) **with respect to** v is:

$$\sum_v u_n = \lim_v u_n = \int_v u_n = \bigcup_{V \in v} \bigcap_{n \in V} u_n$$

(Dolecki, Mynard 2002) **Monotone sequential contours** of rank 1 are exactly Fréchet filters on infinite subsets of ω; If monotone sequential contours off ranks less then α are already defined, than u is a monotone sequential contours of rank α if $u = \sum_{\mathrm{Fr}} u_n$, where $(u_n)_{n<\omega}$ is a discrete sequence of monotone sequential contours such that: $r(u_n) \leq r(u_{n+1})$ and $\lim_{n<\omega}(r(u_n)+1) = \alpha$.

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Define classes $\mathcal{P}_\alpha$ of **P-hierarchy** for $\alpha < \omega_1$ as follows: $u \in \mathcal{P}_\alpha$ if there is no monotone sequential contour c_α of rank α such that $c_\alpha \subset u$, and for each $\beta < \alpha$ there exists a monotone sequential contour c_β of rank β such that $c_\beta \subset u$.

Recall that (Seq, Fr) is a topological space on the set of finite sequences of natural numbers with a maximal topology for which (recursively): if $v \in Seq$ then $\{v^\frown n : n < \omega\} \setminus U(v)$ is finite for each neighborhood $U(v)$ of v.

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P_α-ultrafilters are precisely proper $L_\alpha^{(Seq,Fr)}$-ultrafilters.

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Let (X, τ) be a topological space, we say that (X, τ) is the *Countable - Countable space* (**C-C** space) if for each countable $A \subset X$ if Cantor-Bandixson rank of A is countable, then $\mathrm{cl}(A)$ is also countable.

We say that a topological spaces has the *Convergence Neighborhood Sequence property* (is **CNS**) if for each sequence (x_n) in X, for each sequence (U_n) of neighbourhoods of elements of (x_n) there exist a convergence sequence (y_n) and sequences $(U'_n), (V_n)$ of neighborhoods of elements x_n and y_n respectively, such that (V_n) is pairwise disjoint and there is a sequence (h_n) of homeomorphisms such that $U'_n \subset U$, $h_n : U'_n \to V_n$ and $h_n(x_n) = y_n$.

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Theorem

Let (X, τ) be a topological space, then there exists CNS-extension $CNS(X, \tau)$ of (X, τ) which preserve C-C, non C-C, sequentiality and T_i-axiom for $i \in \{1, 2, 3, 4\}$. Such extension <u>*is not*</u> *unique.*

Remark

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Example

Take ω *and* $\mathbb{A}$ *- a MAD on* ω*. We define topology on* $X_1 = \omega \cup \mathbb{A}$ *by the base of naighborhoods in each point. If* $x \in \omega$ *then* $\{x\}$ *is open, if* $x \in \mathbb{A}$ *then a neighborhood of* x *is* x *with any co-finite set on* $x \subset \omega$*. This is sequential,* T_2 *and non C-C space, which by the previous Theorem may be extend to CNS.*

Theorem (redefined, Baumgartner 1995)

If proper J_{ω^2} ultrafilters exist, then for each countable ordinal α the class of proper $J_{\omega^{\alpha+1}}$-ultrafilters is nonempty.

Theorem

Let (X, τ) be a sequential, C-C, CNS space. If proper $L_2^{(X,\tau)}$ ultrafilter exists, then for each countable ordinal α the class of proper $L_{\alpha+1}^{(X,\tau)}$-ultrafilters is non empty.

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Theorem (Baumgartner, 1995)

Let u be a proper $J_{\omega^{\alpha+2}}$ ultrafilter then there is a function $f:\omega\to\omega$ such that $f(u)$ is a proper J_{ω^2} ultrafilter

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Example

Let X be a disjoint sum of three infinite countable sets X_1, X_2 and X_3. Let τ be a maximal topology for which the base of neighbourhood system has the following properties

1) if $x \in X_1$ then $\{x\}$ is open,

2) if $x \in X_2$ then $X_1 \setminus U(x)$ is finite for each neibouthood $U(x)$ of x,

2) if $x \in X_3$ then $(X_1 \cup X_2) \setminus U(x)$ is finite for each neibouthood $U(x)$ of x.

Note that each free ultrafilter is a proper $L_3^{(X,\tau)}$-ultrafilter.

Theorem (Laflamme 1996) ($\mathrm{MA}_{\sigma-\mathrm{centr}}$)

There is proper $J_{\omega^{\omega+1}}$-ultrafilter all of whose RK-predecessors are proper $J_{\omega^{\omega+1}}$-ultrafilters.

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We say that the class K of ultrafilters is closed under sums if for each sequence (o_n) of filters from the class K, for each filter $o \in K$, the sum $\sum_o o_n$ is in the class K.

Theorem (Baumgartner 1995)

The class of ordinal ultrafiltes is closed under sums

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Questions:

(Baumgartner) If α is limite, is the class of proper J_{ω^α} ultrafilters nonempty(even under some set theoretical assumptiom)?

Is there (under some set theoretical assumption) an element of P-hierarchy which is not an ordinal ultrafilter?

(Baumgartner, later Shelah) Is there a model with no ordinal ultrafilters?

Is there a model with no P-hierarchy?

For which topological spaces proper $L_2^{(X,\tau)}$ ultrafilters are P-points?

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Thank You for your attention