Convergence Measure Spaces

An Approach Towards the Duality Theory of Convergence Groups

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Contents

Pontryagin Duality

Dual group, $\hat{G} = (\mathbb{C}Hom(G, \mathbb{T}), \tau_{co})$

The set of all continuous characters of an **abelian topological** group with operation of pointwise multiplication is called `character group`, and this group with **compact open topology** is called `(Pontryagin) dual group`.

Pontryagin duality

For a topological abelian group there is a natural evaluation homomorphism

$$\alpha_G : G \to \hat{\hat{G}} \qquad \alpha_G(g)(\chi) = \chi(g) \quad \forall\, g \in G.$$

If this evaluation map is a topological isomorphism then the group is said to satisfy `Pontryagin duality` or is said to be `Pontryagin reflexive`.

Pontryagin-van Kampen theorem

Every locally compact abelian (**LCA**) group is Pontryagin reflexive.

Pontryagin Duality (Consequences and Extensions)

Consequences of Pontryagin duality theorem

- Describes the topological or algebraic property of LCA groups in terms of their dual groups.
- It explains why the Pontryagin duality is satisfied in LCA groups.

Extensions

- **Kaplan (1948)** [1]: Pontryagin duality theorem is obtained for the infinite product and direct sum of reflexive groups.
- **Smith (1952)**[2]: Banach spaces as topological groups are Pontryagin reflexive.
- **Butzmann (1977)**[3]: Pontryagin duality is extended to the category of convergence abelian groups.

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[1] Kaplan, S. (1948). Extensions of the Pontryagin duality I: Infinite products. Duke Math J, 15(3):649-658.

[2] Smith, M. F. (1952). The Pontrjagin duality theorem in linear spaces. Ann of Math, (2):248-253.

[3] Butzmann, H.-P. (1977). Pontrjagin-Dualität für topologische Vektorräume. Arch Math (Basel), (28):632–637.

Continuous Convergence Structure

Convergence space

A mapping $\lambda : X \to \mathfrak{F}(X)$ which associates a member of X to the power set of the set of all filters on X is called `convergence structure` on X if the following conditions are satisfied:
(i) $\mathfrak{F}_x \in \lambda(x)$, here $\mathfrak{F}_x$ is the filter generated by x;
(ii) If $\mathfrak{F} \in \lambda(x)$ and $\mathfrak{F} \subset \mathfrak{G}$ then $\mathfrak{G} \in \lambda(x)$;
(iii) If $\mathfrak{F}, \mathfrak{G} \in \lambda(x)$ then there is a filter contained in $\mathfrak{F} \bigcap \mathfrak{G}$ which belongs to $\lambda(x)$.
A convergence space is the pair (X, λ).

Continuous convergence structure, λ_c

The `continuous convergence structure` on the character group of a topological abelian group is the coarsest convergence structure which makes the evaluation mapping $e : \mathbb{C}Hom(G, \mathbb{T}) \times G \to \mathbb{T}$ continuous.

Admissible Topology

Admissible topology on $\mathbb{C}Hom(G, \mathbb{T})$

A topology on $\mathbb{C}Hom(G, \mathbb{T})$ is called `admissible` if the evaluation mapping $e : \mathbb{C}Hom(G, \mathbb{T}) \times G \to \mathbb{T}$, $e(\chi, g) = \chi(x)$ is continuous.

Reflexive Admissible Topological Group[4]

If G is a reflexive topological abelian group, then the evaluation mapping is continuous if and only if G is locally compact.

Remark

$C_c(X)$ and $C_{co}(X)$ denotes the set of continuous real valued functions on a convergence space X with continuous convergence structure and compact open topology respectively and $C_c(X) = C_{co}(X)$ if X is locally compact.

[4] Martin-Peinador,E.(1995). A reflexive admissible topological group must be locally compact. Proc Amer Math Soc, 123(11):3563-3566.

Binz-Butzmann Duality

Binz-Butzmann dual, $\Gamma G = (\mathbb{C}Hom(G, \mathbb{T}), \lambda_c)$

The character group of a topological group with continuous convergence structure is called `Binz-Butzmann dual`[5].

Remark

If G is locally compact convergence group[6] then

$$(\mathbb{C}Hom(G, \mathbb{T}), \tau_{co}) = (\mathbb{C}Hom(G, \mathbb{T}), \lambda_c).$$

Evaluation mapping

For each convergence group G, the mapping $\kappa : G \to \Gamma\Gamma G$ defined by

$$\kappa(g)(\chi) = \chi(g) \; \forall \; g \in G, \;\; \chi \in \Gamma G$$

is a continuous group homomorphism.

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[5] Chasco, M.J. and Martn-Peinador, E.(1994). Binz-Butzmann duality versus Pontryagin duality. Arch Math, 63(3):264-270.

[6] Butzmann, H.-P.(2000). Duality theory for convergence groups. Topology Appl, 111(1):95-104.

Duality in Convergence Groups

c-reflexive groups

A convergence group is `c-reflexive` if κ is an isomorphism.

Remarks

- There exists **non-reflexive locally compact** convergence group.
- There exists **infinite dimensional locally compact convergence vector space** which is reflexive.

Problem

- To study the reflexivity in convergence groups.
    - Characterise the class of **reflexive locally compact convergence groups**.

Convergence Space (Open and Closed Sets)

Open and closed sets in convergence spaces

- Filter $\mathcal{N}(x) = \bigcap\{\mathfrak{F} : \mathfrak{F} \in \lambda(x)\}$ is called `neighbourhood filter` of x and its elements `neighbourhoods` of x. A set $U \subset X$ is open if it is neighbourhood of each of its points.
- For each $A \in X$ the `adherence` of A is the set $a(A) = \{x \in X :$ there is $\mathcal{F} \in \lambda(x)$ such that $A \in \mathcal{F}\}$ and $A \subset X$ is `closed` if $a(A) = A$.

Remarks

- In general adherence operator need not be idempotent.
- Neighbourhood filter of a point need not convergence to that point.

Topological convergence structure

The `topological convergence structure` λ on a topological space (X, τ) is defined as $\mathcal{F} \in \lambda(x)$ if and only if $\mathfrak{U}_x \subset \mathcal{F}$, here $\mathfrak{U}_x$ is the set of all topological neighbourhoods of x.

Convergence Space (Topological Modification)

Topological convergence

A convergence space is topological iff it has the topological convergence structure.

Example of a compact non-topological convergence space[7]

The ultrafilter modification of $[0,1]$ (the finest convergence on $[0,1]$ that has the same convergent ultrafilter as the usual topology of $[0,1]$) is a compact Hausdorff convergence space which is not topological.

Topological Modification

- A topology can be associated to every convergence space, called topological modification (denoted, $o(X)$) of the convergence space.
- The collection of all open sets satisfy the axioms of a topology.
- For a convergence space (X, λ) we denote this topology as λ_{tm}.

[7] Beattie,R. and Butzmann,H.-P.(2013). Convergence Structures and Applications to Functional Analysis. Bücher. Springer Netherlands, 2013.

Convergence Spaces With Same Topological Modification

Other convergence structures

A convergence space is

- `Pre-topological` if neighbourhood filter of each point converges to that point.
    - `Pre-topological modification`, $\pi(X)$ associated to a convergence space is defined as $\mathfrak{F} \in \lambda(x)$ in $\pi(X)$ if and only if $\mathfrak{F} \supseteq \mathcal{N}(x)$
- `Choquet` if $\mathfrak{F} \in \lambda(x)$ in X whenever every ultrafilter finer then $\mathfrak{F}$ converges to x in X.
    - `Choquet modification`, $\chi(X)$ associated to a convergence space is defined as $\mathfrak{F} \in \lambda(x)$ in $\chi(X)$ if $\mathfrak{G} \in \lambda(x)$ in X for every ultrafilter $\mathfrak{G}$ on X finer than $\mathfrak{F}$.

Example

Consider a convergence space X which is not Choquet. X and $\chi(X)$ need not be homeomorphic but the topological modification of X and $\chi(X)$ are same.

Convergence Measure Space

Convergence measure space

A convergence measure space is a quadruple $(X, \lambda, \mathcal{M}, \mu)$, where $(X, \mathcal{M}, \mu)$ is a measure space and (X, λ) is a convergence space such that $\lambda_{tm} \subset \mathcal{M}$, i.e every open set (in the sense of convergence) is measurable.

Theorem

The topological modification (X, λ_{tm}) of a compact convergence space (X, λ) is always a compact topological convergence space.

Remarks

- Topological k-spaces are the topological modification of the locally compact convergence spaces[8].
- It is not trivial to extend the theory from topological measure spaces.

[8] Kent,D. and Richardson,G. (1976). Locally compact convergence spaces. Mich Math J, 22(4):353-360, 1976.

Representation by Linear Functional

Theorem: Riesz-Markov theorem-I

Let (X, λ) be a convergence space whose topological modification is locally compact topological space (X, λ_{tm}) and $I : C_c(X) \to \mathbb{R}$ a continuous, positive linear map. Then, there is a uniquely determined Radon measure μ with compact support such that $I(f) = \int f d\mu \, \forall \, f \in C(X)$.

Proof of Riesz-Markov theorem for locally compact topological spaces

- One point compactification of local compact topological space.
- Urysohn lemma and Tietz extension theorem.
- We do not know whether this theorem can be extended to the class of locally compact convergence spaces.

Problem

To characterise the convergence spaces whose topological modification is locally compact.

Representation by Linear Functionals-II

Theorem

A convergence space whose topological modification is locally compact topological space must be a locally compact space.

Theorem: Riesz-Markov theorem-II

Let (X, λ) be a convergence space whose topological modification is locally compact topological space (X, λ_{tm}) and $I : C_{co}(X) \to \mathbb{R}$ a continuous, positive linear functional with compact support. Then, there is a uniquely determined Radon measure μ such that $I(f) = \int f d\mu \; \forall \; f \in C(X)$.

Remarks

- Topological modification of a convergence group need not be a topological group.

Duality Theory of Convergence Groups

Problem

- To extend the definition of Haar measure for a class of non-topological convergence groups.
- To characterise those convergence groups whose topological modification is a locally compact topological group.
- Are the convergence groups whose topological modification locally compact topological group, reflexive?

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[2] E. Binz. *Continuous convergence on C(X)*. Lecture notes in mathematics. Springer-Verlag, 1975.

[3] M. Bruguera and M. J. Chasco. Strong reflexivity of abelian groups. *Czechoslovak Math J*, 51(1):213–224, 2001.

[4] H.-P. Butzmann. Pontrjagin-dualität für topologische vektorräume. *Arch Math (Basel)*, 28(1):632–637, 1977.

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[6] M. J. Chasco and E. Martín-Peinador. Binz-Butzmann duality versus Pontryagin duality. *Arch Math (Basel)*, 63(3):264–270, 1994.

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[8] D. Kent and G. Richardson. Locally compact convergence spaces. *Mich Math J*, 22(4):353–360, 1976.

[9] E. Martín Peinador. A reflexive admissible topological group must be locally compact. *Proc Amer Math Soc*, 123(11):3563–3566, 1995.

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