Maximal Homogeneous Spaces

Evgenii Reznichenko
Moscow State University
erezn@inbox.ru

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CONTENTS

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A space X is maximal homogeneous iff X is a maximal homogeneous subspace of βX containing X. Clearly, $H(X)$ is β-stable and maximal homogeneous if X is homogeneous.

β-stable spaces

Proposition

If X is a first countable space then X is β-stable.

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(T.Banach, R.) Suppose that X is a homogeneous realcompact space and there exists a convergent sequence in X. Then X is maximal homogeneous.

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We say that a space X is **totally countably** p-**compact** if, for any infinite $M \subset X$, there exists an infinite $L \subset M$ such that any sequence $(x_n)_{n\in\omega} \subset L$ ($x_i \neq x_j$ for $i \neq j$) has a p-limit in X.

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Theorem

If $p \in \omega^*$ and X is a totally countably p-compact space, then X^ω is totally countably p-compact and, hence, countably compact.

Maximal homogeneous extremally disconnected space

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Let X be a maximal homogeneous extremally disconnected space. If X contains a nonclosed dicrete sequence of points, then X is totally countably p-compact for some $p \in \omega^*$.

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Corollary

Let X be a homogeneous extremally disconnected space. If X contains a nonclosed dicrete sequence of points, then $H(X)$ is maximal homogeneous, extremally disconnected, and totally countably p-compact for some $p \in \omega^*$.

Maximal homogeneous groups

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The answer is "no" under CH. Under CH there exists an extremally disconnected group G containing a nonclosed dicrete sequence of points. $H(G)$ is an extremally disconnected countably compact space.
Any extremally disconnected countably compact group is finite.

Products of groups

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(Hart and van Mill, 1991) ($MA_{countable}$) There exists a countably compact group whose square is not countably compact.

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There exists homogeneous spaces X and Y such that

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