On the center of distances

by

Wojciech Bielas, Szymon Plewik (the speaker) and Marta Walczyńska

(Katowice - July 2016)

Abstract from the version, which was posted on the website arXiv:1605.03608.

We introduce the notion of the center of distances of a metric space, which is required for a generalization of the theorem by J. von Neumann about permutations of two sequences with the same set of cluster points in a compact metric space. Also, the introduced notion is used to study sets of subsums of some sequences of positive reals, as well for some impossibility proofs. We compute the enter of distances of the Cantorval, which is the set of subsums of the sequence $\frac{3}{4}, \frac{1}{2}, \frac{3}{16}, \frac{1}{8}, \dots, \frac{3}{4^n}, \frac{2}{4^n}, \dots$, and also for some related subsets of the reals.

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Center of distances: Definition

Given a metric space X with the distance d, consider the set

$$S(X) = \{\alpha \colon \forall_{x \in X} \exists_{y \in X} d(x,y) = \alpha\},$$

which will be called the *center of distances* of X.

Obviously: If X is an interval $[a,b]$, then $S([a,b]) = [0, \frac{b-a}{2}]$; Also $S(\mathbb{Q}) = \mathbb{Q}$, whenever $\mathbb{Q}$ is the sets of all rationals; But $S(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$. In fact, the computation of the center of distances— even for many well-known subsets of the reals—is not an easy task because it requires skillful use of fractions.

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Generalization of a theorem by J. von Neumann

Theorem

Suppose that sequences $\{a_n\}_{n\in\omega}$ and $\{b_n\}_{n\in\omega}$ have the same set of cluster points $C \subseteq X$, where (X, d) is a compact metric space. If $\alpha \in S(C)$, then there exists a permutation $\pi\colon \omega \to \omega$ such that $\lim_{n\to+\infty} d(a_n, b_{\pi(n)}) = \alpha$.

Some proofs, but for $\alpha = 0$, were given by J. von Neumann (1935), P. R. Halmos (1968) or J. A. Yorke (1969). But the back-and-forth method, which was developed by E. V. Huntington (1905 and 1917), shows that the center of distances is intuitive and natural.

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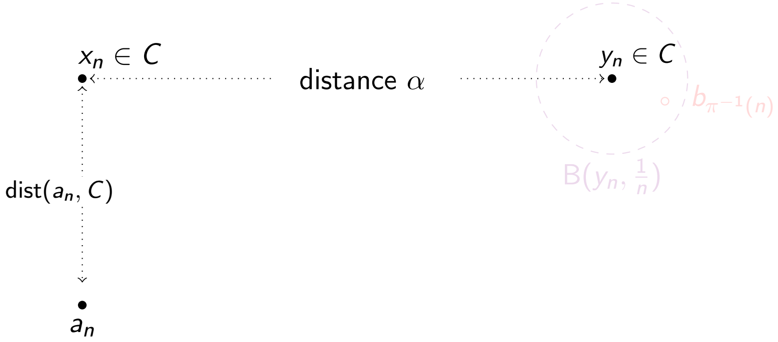

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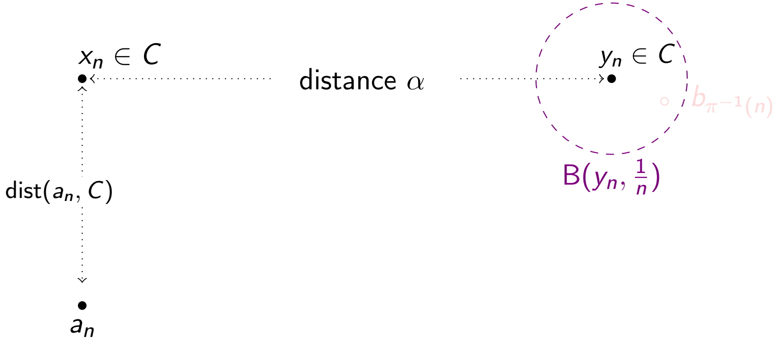

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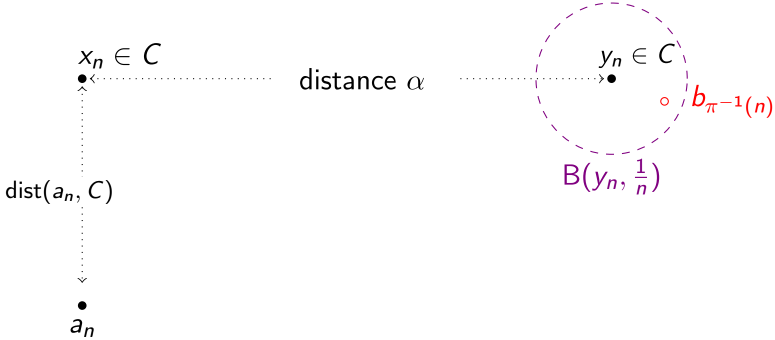

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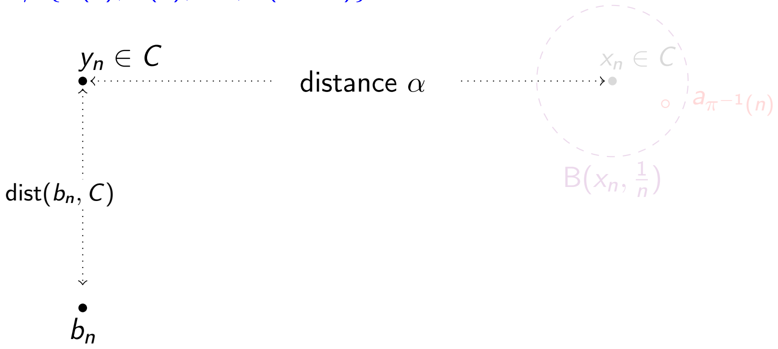

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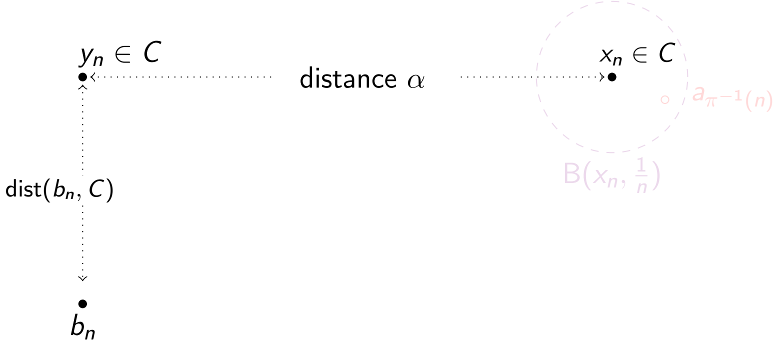

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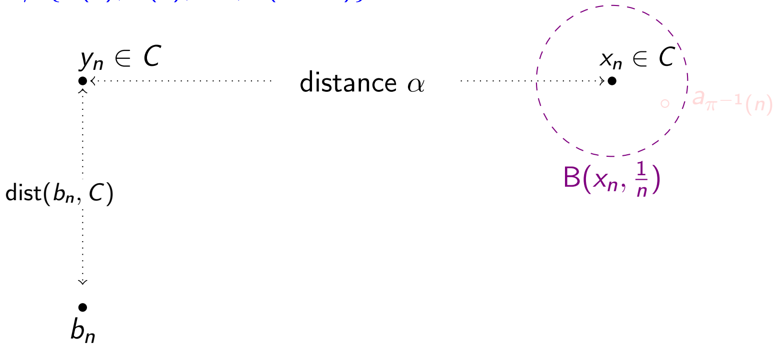

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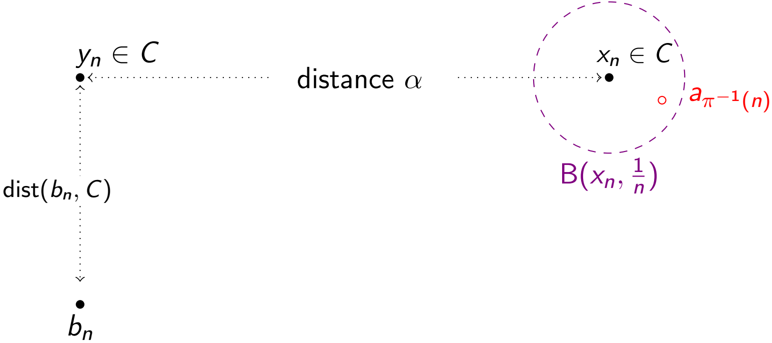

Sets of subsums: achievement sets

If $\{a_n \colon n \in \omega\}$ is a sequence of reals, then the set

$$X = \{\sum_{n \in A} a_n \colon A \subseteq \omega\}$$

is called the *set of subsums* of the sequence $\{a_n\}$ (Some authors used the name *achievement set*.)

Proposition

If X is the set of subsums of a sequence $\{a_n\}_{n \in \omega}$, then $a_n \in S(X)$, for all $n \in \omega$.

Proof.

Suppose $x = \sum_{n \in A} a_n \in X$. If $n \in A$, then $x - a_n \in X$ and $d(x, x - a_n) = a_n$. When $n \notin A$, then $x + a_n \in X$ and $d(x, x + a_n) = a_n$. □

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The center of distances of the set of subsums of a given sequence can sometimes easily be determined. For example, the unit interval is the set of subsums of the sequence $\{\frac{1}{2^n}\}_{n>0}$. So, $[0, \frac{1}{2}]$ is the center of distances in this case. The following theorem works also for the for the Cantor ternary set, if $q = 3$ and $a = 2$.

Theorem

If $q > 2$ and $a \geqslant 0$, then the center of distances of the set of subsums of a geometric sequence $\{\frac{a}{q^n}\}_{n>0}$ consists of exactly the terms of the sequence $0, \frac{a}{q}, \frac{a}{q^2}, \ldots$.

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An example of a Cantorval

Following A. Guthrie and J. E. Nymann (1988), consider the set of subsums

$$\mathbb{X} = \left\{ \sum_{n>0} \frac{x_n}{4^n} : \forall_n \, x_n \in \{0, 2, 3, 5\} \right\}.$$

Thus, $\mathbb{X} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathcal{C}_1 = \{\sum_{n \in A} \frac{2}{4^n} : 0 \notin A \subseteq \omega\}$ and $\mathcal{C}_2 = \{\sum_{n \in B} \frac{3}{4^n} : 0 \notin B \subseteq \omega\}$. Following P. Mendes and F. Oliveira (1994), because of its topological structure, one can call this set a *Cantorval* (or an $\mathcal{M}$-*Cantorval*). In fact, each Cantorval is a regularly closed subset of the reals whit the boundary being a homeomorphic copy of the Cantor set. An approximation of the Cantorval $\mathbb{X} \subset [0, \frac{5}{3}]$:

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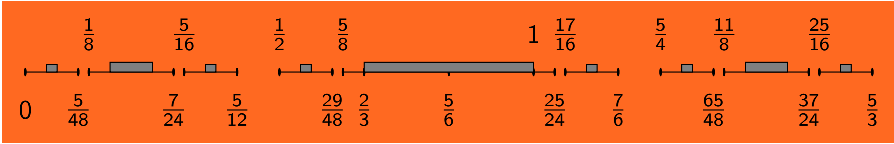

$\mathbb{X}$-intervals and $\mathbb{X}$-gaps

If $\mathbb{Y}$ is a closed subset of the reals, then any maximal interval (α, β) disjoint from $\mathbb{Y}$ is called an $\mathbb{Y}$-*gap*, but a maximal interval $[\alpha, \beta]$ included in $\mathbb{Y}$ is called an $\mathbb{Y}$-*interval*.

Thus, the interval $[\frac{2}{3}, 1]$ is the longest $\mathbb{X}$-interval (for a proof see Corollary 6 in arXiv:1605.03608). But intervals $(\frac{5}{12}, \frac{1}{2}) = (\sum_{n=2} \frac{5}{4^n}, \frac{1}{2})$ and $(\frac{7}{6}, \frac{5}{4}) = (\sum_{n=1} \frac{3}{4^n} + \sum_{n=2} \frac{2}{4^n}, \frac{1}{2} + \frac{3}{4})$ are the longest $\mathbb{X}$-gaps. Six $\mathbb{X}$-gaps:

$(\frac{5}{48}, \frac{1}{8}) = (\sum_{n=3} \frac{5}{4^n}, \frac{2}{16})$,
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$(\frac{7}{24}, \frac{5}{16}) = (\frac{3}{16} + \sum_{n=3} \frac{5}{4^n}, \frac{3}{16} + \frac{2}{16}) = (\frac{3}{16} + \frac{5}{48}, \frac{3}{16} + \frac{1}{8})$,
$(\frac{29}{48}, \frac{5}{8}) = (\frac{1}{2} + \sum_{n=3} \frac{5}{4^n}, \frac{1}{2} + \frac{2}{16}) = (\frac{1}{2} + \frac{5}{48}, \frac{1}{2} + \frac{1}{8})$,
$(\frac{25}{24}, \frac{17}{16}) = (\frac{3}{4} + \frac{3}{16} + \sum_{n=3} \frac{5}{4^n}, \frac{3}{4} + \frac{3}{16} + \frac{2}{16}) = (\frac{3}{4} + \frac{3}{16} + \frac{5}{48}, \frac{3}{4} + \frac{5}{16})$
$(\frac{65}{48}, \frac{11}{8}) = (\frac{3}{4} + \frac{1}{2} + \sum_{n=3} \frac{5}{4^n}, \frac{3}{4} + \frac{1}{2} + \frac{2}{16}) = (\frac{3}{4} + \frac{1}{2} + \frac{5}{48}, \frac{5}{4} + \frac{1}{8})$
$(\frac{37}{24}, \frac{25}{16}) = (\frac{3}{4} + \frac{1}{2} + \frac{3}{16} + \sum_{n=5} \frac{5}{4^n}, \frac{5}{4} + \frac{5}{16}) = (\frac{5}{4} + \frac{3}{16} = \frac{5}{48}, \frac{5}{4} + \frac{5}{16})$;
have the length $\frac{1}{48}$. To see these use the formula for the sum of an infinite geometric series and rules of adding fractions!

If $\mathbb{Y}$ is a closed subset of the reals, then any maximal interval (α, β) disjoint from $\mathbb{Y}$ is called an $\mathbb{Y}$-*gap*, but a maximal interval $[\alpha, \beta]$ included in $\mathbb{Y}$ is called an $\mathbb{Y}$-*interval*.
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$\mathbb{X}$-intervals and $\mathbb{X}$-gaps

If $\mathbb{Y}$ is a closed subset of the reals, then any maximal interval (α,β) disjoint from $\mathbb{Y}$ is called an $\mathbb{Y}$-*gap*, but a maximal interval $[\alpha,\beta]$ included in $\mathbb{Y}$ is called an $\mathbb{Y}$-*interval*.
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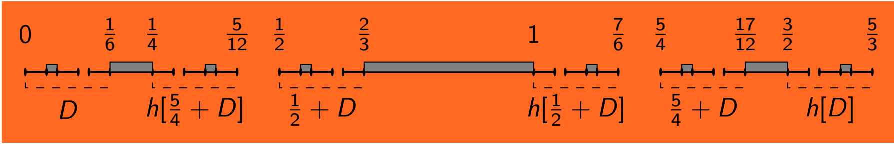
have the length $\frac{1}{48}$. To see these use the formula for the sum of an infinite geometric series and rules of adding fractions!

Affine properties of $\mathbb{X}$-intervals

To determine $\mathbb{X}$-intervals examine the affine properties of $\mathbb{X}$. First of all:
The involution $h\colon \mathbb{X} \to \mathbb{X}$ defined by the formula

$$x \mapsto h(x) = \frac{5}{3} - x$$

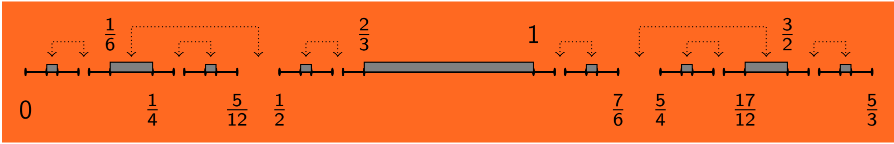
is the symmetry of $\mathbb{X}$ with respect to the point $\frac{5}{6}$. So, we get $\mathbb{X} = \frac{5}{3} - \mathbb{X}$ and $\mathbb{X} = h[\mathbb{X}]$.
Since $\frac{1}{4} \cdot \mathbb{X} = [0, \frac{5}{12}] \cap \mathbb{X}$, we can deduce that $[\frac{1}{6}, \frac{1}{4}]$ and $[\frac{12}{17}, \frac{3}{2}]$ are $\mathbb{X}$-intervals of the length $\frac{1}{12}$, which equals to the maximal length of $\mathbb{X}$-gaps. But if $D = [0, \frac{1}{6}] \cap \mathbb{X}$, then we get the similarities as shown below.



Corollary

The Cantorval $\mathbb{X} \subset [0, \frac{5}{3}]$ has Lebesgue measure 1, but it's boundary has Lebesgue measure 0.

Proof. There exists a one-to-one correspondence between $\mathbb{X}$-gaps and $\mathbb{X}$-interval as it is shown below.



So, we calculate the sum of lengths of all $\mathbb{X}$-gaps as follows:

$$\frac{1}{6} + 6 \cdot \frac{1}{3 \cdot 4^2} + \frac{1}{8} \cdot \frac{3}{4} + \ldots + \frac{1}{8} \cdot \left(\frac{3}{4}\right)^n + \ldots = \frac{1}{6} + \frac{1}{8} \sum_{n \geqslant 0} \left(\frac{3}{4}\right)^n = \frac{2}{3}.$$

Thus, the sum of lengths of all $\mathbb{X}$-intervals is one-third greater than the previous sum, i.e., it equals 1. $\square$

The center of distances $S(\mathbb{X})$

Theorem

The center of distances of the Cantorval $\mathbb{X}$ consists of exactly the terms of the sequence $0, \frac{3}{4}, \frac{1}{2}, \ldots, \frac{3}{4^n}, \frac{2}{4^n}, \ldots$.

Proof.

Move a $\mathbb{X}$- gap (p,q) to the left by α searching for a point $x \in \mathbb{X}$ such that $\alpha - x < 0$ and $p < \alpha + x < q$. After several attempts, find recursive formulas that allow you to find $\alpha \notin S(\mathbb{X})$; See arXiv:1605.03608 for details. □

The center of distances $S([0, \frac{5}{3}] \setminus \operatorname{Int} \mathbb{X})$

Theorem

The center of distances of the set $[0, \frac{5}{3}] \setminus \operatorname{Int} \mathbb{X}$ *is trivial, i.e.,* $S([0, \frac{5}{3}] \setminus \operatorname{Int} \mathbb{X}) = \{0\}$.

Proof.

Apply a method of the proof of the previous theorem. $\square$

Let us add that the set of subsums of the sequence $\{\frac{1}{4^n}\}_{n\in\omega}$ is included in $\mathbb{X} \setminus \operatorname{Int} \mathbb{X}$. One can check this, observing that each number $\sum_{n\in A} \frac{1}{4^n}$, where the nonempty set $A \subset \omega$ is finite, is the right end of an $\mathbb{X}$-interval.

Impossibility

Theorem

$S(\mathbb{X} \setminus \operatorname{Int} \mathbb{X}) = \{0\} \cup \{\frac{1}{4^n} : n \in \omega\}$, i.e., the center of distances of the set $\mathbb{X} \setminus \operatorname{Int} \mathbb{X}$ consists of exactly of the terms of the sequence $0, \frac{1}{4}, \frac{1}{16}, \ldots, \frac{1}{4^n}, \ldots$.

Proof.

Use the numbers $0, \frac{1}{4}, \frac{1}{3} = \sum_{n>0} \frac{5}{4^{2n}}, \frac{1}{2}, \frac{17}{32}$ and 1 which are in $\mathbb{X} \setminus \operatorname{Int} \mathbb{X}$. Then check that number $\frac{1}{4^n}$, where $0 < 1$, belong to $S(\mathbb{X} \setminus \operatorname{Int} \mathbb{X})$. $\square$

Corollary

Neither the set $[0, \frac{5}{3}] \setminus \operatorname{Int} \mathbb{X}$ nor the set $\mathbb{X} \setminus \operatorname{Int} \mathbb{X}$ is the set of subsums of a sequence. $\square$

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Use the numbers $0, \frac{1}{4}, \frac{1}{3} = \sum_{n>0} \frac{5}{4^{2n}}, \frac{1}{2}, \frac{17}{32}$ and 1 which are in $\mathbb{X} \setminus \operatorname{Int} \mathbb{X}$. Then check that number $\frac{1}{4^n}$, where $0 < 1$, belong to $S(\mathbb{X} \setminus \operatorname{Int} \mathbb{X})$. □

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Digital representation of points in the Cantorval $\mathbb{X}$

Sequences $\{a_n\}_{n>0}$ and $\{b_n\}_{n>0}$ are digital representations of a point $x \in \mathbb{X}$, whenever $a_n, b_n \in \{0, 2, 3, 5\}$ and $\sum_{n>0} \frac{a_n}{4^n} = \sum_{n>0} \frac{b_n}{4^n} = x$.

Theorem

Assume that $x \in \mathbb{X}$ has more than one digital representation. There exists the finite or infinite sequence of positive natural numbers $n_0 < n_1 < \ldots$ and exactly two digital representations $\{a_n\}_{n>0}$ and $\{b_n\}_{n>0}$ of x such that:

- $a_k = b_k$, *as far as* $0 < k < n_0$;
- $a_{n_0} = 2$ *and* $b_{n_0} = 3$;
- $a_{n_k} = 5$ *and* $b_{n_k} = 0$, *for odd* k;
- $a_{n_k} = 0$ *and* $b_{n_k} = 5$, *for even* $k > 0$;
- $a_i \in \{3, 5\}$ *and* $a_i - b_i = 3$, *as far as* $n_{2k} < i < n_{2k+1}$;
- $a_i \in \{0, 2\}$ *and* $b_i - a_i = 3$, *as far as* $n_{2k+1} < i < n_{2k+2}$.

Corollary

If $x \in \mathbb{X}$ has a digital representation $\{x_n\}_{n>0}$ such that $x_n = 2$ and $x_{n+1} = 3$ for infinitely many n, then this representation is unique.

Corollary

Let $A \subset \{\frac{2}{4^n} : n > 0\} \cup \{\frac{3}{4^n} : n > 0\} = B$ be such that $B \setminus A$ and A are infinite. Then the set of subsums of a sequence consisting of different elements of A is homeomorphic to the Cantor set.

Proof.

Correct typing missprints in arXiv:1605.03608. □

Thank you for your attention