Normal spanning trees in uncountable graphs, and almost disjoint families

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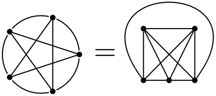
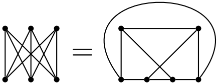
29 July 2016

Characterising properties by forbidden substructures

Some examples involving planarity

- *Kuratowski's Theorem ('30):* A finite graph is planar if and only if it doesn't embed K_5 and $K_{3,3}$.

$K_5 =$ 

$K_{3,3} =$ 

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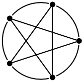
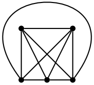
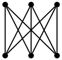
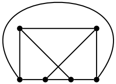
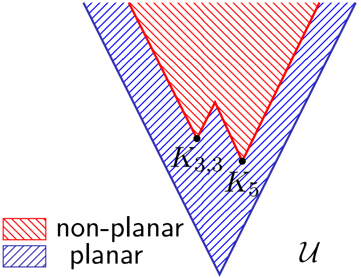
$K_5 =$  $=$ 

$K_{3,3} =$  $=$ 



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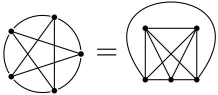
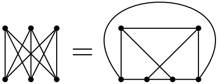
$K_5 =$ 

$K_{3,3} =$ 

- *Claytor's Theorem ('34):* A Peano continuum is planar if and only if it doesn't embed K_5, $K_{3,3}$, L_5 and $L_{3,3}$.

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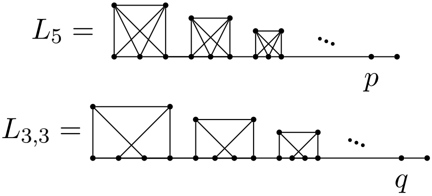
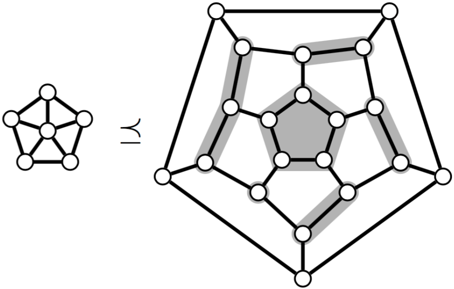


The Graph-Minor Theorem

The graph-theoretic notion of a minor

Say that $G \preccurlyeq H$ (*G is a minor of H*) if G embeds into a monotone quotient of H.



Alternative description: G can be obtained by deleting and contracting some edges of H.

The Graph-Minor Theorem

Describing properties by forbidding *finitely many* substructures

Graph-Minor Theorem (Robertson & Seymour, '83-'04, GM I–XX)

Any property of finite graphs that is preserved under taking minors is characterised by finitely many forbidden minors.

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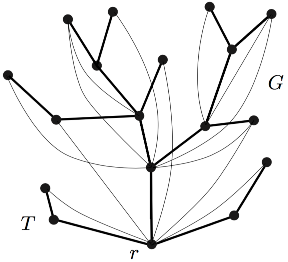
Any property of finite graphs that is preserved under taking minors is characterised by finitely many forbidden minors.

- False for graphs of size $\mathfrak{c}$ (Thomas, '88).
- Open for countable graphs.
- Algorithmic aspects: Checking whether a fixed graph is a minor can be done in polynomial time $\Rightarrow$ all minor-closed properties can be verified in polynomial time.
- Embeddability into a fixed surface (e.g. a torus) is minor-closed. Have to forbid at least 16,000 graphs.

Normal spanning trees (NST)

A generalisation of depth-first-search trees



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- Edges of G grow parallel to branches on the tree T.

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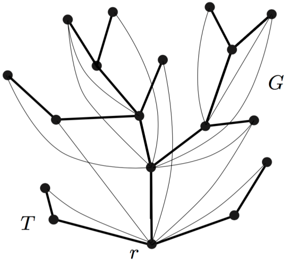

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- Finite connected graphs have NSTs (depth-first search).
- Countable connected graphs have NSTs (Jung, '67).
- Uncountable graphs need not have an NST.
- Having an NST is closed under taking (connected) minors (Jung, '67).

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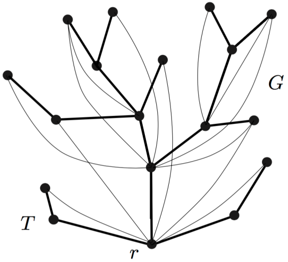

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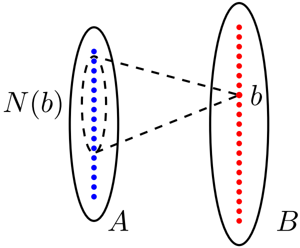
- Uncountable graphs need not have an NST.
- Having an NST is closed under taking (connected) minors (Jung, '67). $\Rightarrow$ What are the (minimal) forbidden minors?

Forbidden substructures for NSTs

Halin's $(\aleph_0, \aleph_1)$-graphs without a normal spanning tree

An $(\aleph_0, \aleph_1)$-graph is bipartite on vertex sets A and B, such that

- $|A| = \aleph_0$,
- $|B| = \aleph_1$, and
- for all $b \in B$, $|N(b)| = \aleph_0$.



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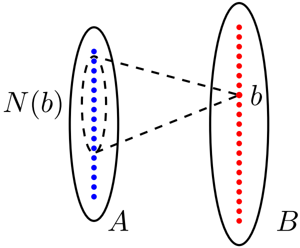

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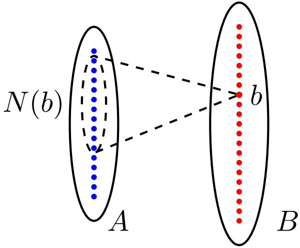

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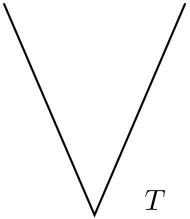

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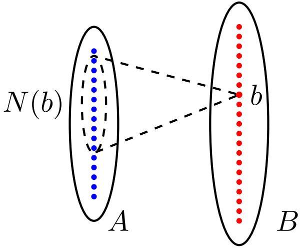

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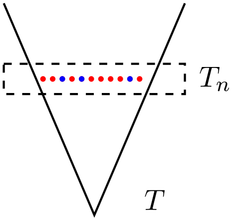

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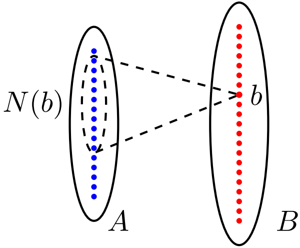

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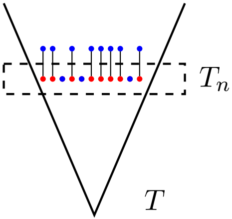

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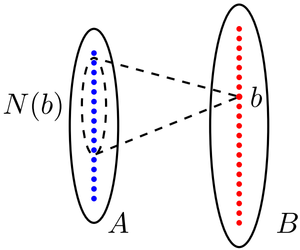

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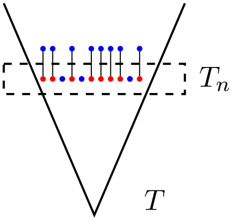
4. so $A \cap T_{n+1}$ is uncountable, contradiction.



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A characterisation due to Diestel and Leader

NST Forbidden Minor Theorem (Diestel & Leader, '01)

A connected graph has an NST if and only if it does not contain an $(\aleph_0, \aleph_1)$-graph or an Aronzsajn tree-graph as a minor.

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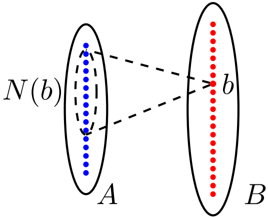
- Open problem (Diestel & Leader): Give a description of the minor-minimal elements of the class of $(\aleph_0, \aleph_1)$-graphs.

- Encode $(\aleph_0, \aleph_1)$-graphs as (multi-)set $\mathcal{N} = \langle N(b_\alpha) \colon \alpha < \omega_1 \rangle$ of ∞-sets $\subset \mathbb{N}$.
- $\Rightarrow$ combinatorics of uncountable collections $\mathcal{N} \subseteq [\omega]^\omega$.
- E.g. consider *Almost disjoint* $(\aleph_0, \aleph_1)$-graphs ($\Leftrightarrow \mathcal{N}$ ADF).



Almost disjoint $(\aleph_0, \aleph_1)$-graphs

For the minor minimal graphs, can restrict our attention to AD-graphs

An $(\aleph_0, \aleph_1)$-graph is *AD* if $|N(b) \cap N(b')| < \infty$ for all $b \neq b' \in B$.

Theorem (Bowler, Geschke, Pitz)

Every $(\aleph_0, \aleph_1)$-graph contains an AD-$(\aleph_0, \aleph_1)$-subgraph.

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- Every collection $\mathcal{N} \subseteq [\omega]^\omega$ of size $< \mathfrak{c}$ has an almost disjoint refinement, i.e. for every $N \in \mathcal{N}$ can pick infinite $N' \subset N$ such that $\{N' \colon N \in \mathcal{N}\}$ is almost disjoint (Baumgartner, Hajnal & Mate, '73; Hechler, '78).
- Best possible, as $\mathcal{N} = [\omega]^\omega$ doesn't have an AD refinement.

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- Best possible, as $\mathcal{N} = [\omega]^\omega$ doesn't have an AD refinement.
- So under $\neg$CH, the theorem follows immediately from Hechler's result. But under CH, one has to find a workaround: Deal with ω_1-towers separately.

Special types of AD-$(\aleph_0, \aleph_1)$-graphs

An overview of $(\aleph_0, \aleph_1)$-graphs with various different combinatorical properties

Graph-theoretic perspective (Diestel & Leader):

- (full) T_2^{tops}: Ctble binary tree, pick branches $\{b_\alpha \colon \alpha < \omega_1\}$. Neighbourhoods are infinite sets $N(b_\alpha) \subset b_\alpha$ ($N(b_\alpha) = b_\alpha$)

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- (weak) tree-family: As T_2^{tops}, but $N(b_\alpha) =^* b_\alpha$ $(N(b_\alpha) \subseteq^* b_\alpha)$
- hidden tree-family: $\mathcal{A}$ is h.t.f. if for some binary tree T, $\{T \cap a \colon a \in \mathcal{A}\}$ a weak tree family

A Martin's Axiom result

Under MA, the (full)-binary trees with tops form a minimal class of $(\aleph_0, \aleph_1)$-graphs

Theorem (Bowler, Geschke, Pitz)

Under MA+$\neg$CH, every $(\aleph_0, \aleph_1)$-graph contains a full T_2^{tops} as subgraph.

- Reminiscent of the result that under MA+$\neg$CH, every ADF of size $< \mathfrak{c}$ is a hidden tree-family (Velickovic '93, Roitman & Soukup '98)
- Proof idea for T_2^{tops}: For every finite subset $B' \subset B$ there are arbitarily large finite trees $\subset A$ with branches being large subsets of B'... Δ-system lemma gives ccc.
- Proof idea for full T_2^{tops}: Take a finite support product.

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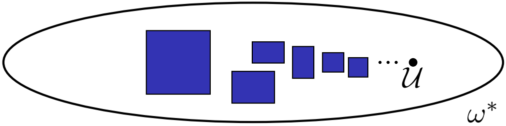
- $\mathcal{U}$-indivisible: For $\mathcal{U} \in \omega^*$ with $\chi(\mathcal{U}) = \omega_1$, pick $N(b_\alpha)$ s.t. $N(b_\alpha)^*$ has $\mathcal{U}$ as unique complete accumulation point.



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- anti-Luzin: $\mathcal{A}$ is a.L. if for all uncountable $\mathcal{B} \subset \mathcal{A}$ there are uncountable $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{B}$ such that $\bigcup \mathcal{C} \cap \bigcup \mathcal{D}$ is finite

Chaos under CH

There are minor-inequivalent classes besides T_2^{tops}

Theorem (Diestel & Leader, '01)

1. Every $(\aleph_0, \aleph_1)$-minor of a T_2^{tops} is divisible
2. Every $(\aleph_0, \aleph_1)$-minor of an indivisible graph is indivisible
3. $\Rightarrow$ under CH (or $\mathfrak{u} = \omega_1$), there are at least two minor-minimal classes of $(\aleph_0, \aleph_1)$-graphs

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There are minor-inequivalent classes besides T_2^{tops}

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- Under CH, there is an $(\aleph_0, \aleph_1)$-graph which contains neither indivisible subgraphs nor T_2^{tops} as a subgraph (Bowler, Geschke & Pitz)

More on indivisible $(\aleph_0, \aleph_1)$-graphs

Different ultrafilters $\leftrightarrow$ different indivisible graphs?

- (Diestel & Leader, '01) If (A, B) and (A', B') are $\mathcal{U}$- and $\mathcal{U}'$-indivisible with $(A, B) \preceq (A', B')$ then $\mathcal{U} \leq_{RK} \mathcal{U}'$.

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[CH]. For every $\mathcal{U}$-indivisible $(\aleph_0, \aleph_1)$ graph G there is an $\mathcal{U}$-indivisible $(\aleph_0, \aleph_1)$ graph H such that $G \not\preceq H$.

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- On first sight, it seems difficult to diagonalise against *all* possible minors, as there are 2^{ω_1} many potential quotients.
- Solution: Only those branching sets that intersect the countable A-side are of importance...

Open questions

Problems I would like to find an answer to:

1. Under CH ($+$ any assumption you like) construct an AD-$(\aleph_0, \aleph_1)$-graph which is minor-incomparable to both indivisible graphs and T_2^{tops} graphs.

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3. Under MA$+\neg$CH, is there a minor-minimal T_2^{tops}?