Hereditary coreflective subcategories in categories of semitopological groups

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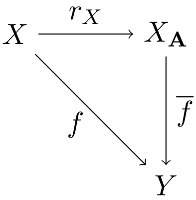
Introduction

Structure	Group operation	Inverse
semitopological group	separately continuous	–
quasitopological group	separately continuous	continuous
paratopological group	continuous	–
topological group	continuous	continuous

Reflective subcategories

Definition

A subcategory $\mathbf{A}$ of $\mathbf{C}$ is reflective in $\mathbf{C}$ provided that for every $X \in \mathbf{C}$ there exists an $\mathbf{A}$-reflection: $X_{\mathbf{A}} \in \mathbf{A}$ and a $\mathbf{C}$-morphism $r_X : X \to X_{\mathbf{A}}$ such that for every $\mathbf{C}$-morphism $f : X \to Y$ where $Y \in \mathbf{A}$ there exists a unique $\mathbf{A}$-morphism $\bar{f} : X_{\mathbf{A}} \to Y$, such that the following diagram commutes:



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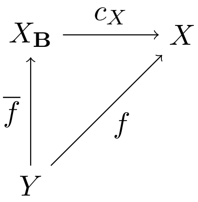
      e.g. **STopAb**, the category of all torsion-free semitopological groups
    - epireflective, closed under the formation of (usual) quotients
      e.g. **QTopGr**, **PTopGr**, **TopGr**, **TopAb**

Coreflective subcategories

Definition

A subcategory $\mathbf{B}$ of $\mathbf{A}$ is coreflective in $\mathbf{A}$ provided that for every $X \in \mathbf{A}$ there exists a $\mathbf{B}$-coreflection: $X_{\mathbf{B}} \in \mathbf{B}$ and an $\mathbf{A}$-morphism $c_X : X_{\mathbf{B}} \to X$ such that for every $\mathbf{A}$-morphism $f : Y \to X$ where $Y \in \mathbf{B}$ there exists a unique $\mathbf{B}$-morphism $\bar{f} : Y \to X_{\mathbf{B}}$, such that the following diagram commutes:



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- **A** denotes an epireflective subcategory of **STopGr**

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Coreflective subcategories in A

- **A** denotes an epireflective subcategory of **STopGr**
- hereditary coreflective ⇒ monocoreflective
- monocoreflective ⇔ closed under the formation of coproducts and extremal quotients
- bicoreflective ⇔ monocoreflective, contains $r_{\mathbf{A}}(\mathbb{Z})$
  e.g. **QTopGr** in **STopGr**, **TopGr** in **PTopGr**

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- in **STopAb**, **QTopAb**: the direct sum with the cross topology
- in **PTopAb**, **TopAb**: the direct sum with the usual topology

Questions

1. What is the hereditary coreflective hull of subcategories of $\mathbf{A}$?

2. Which hereditary coreflective subcategories of $\mathbf{A}$ are bicoreflective in $\mathbf{A}$?

The hereditary coreflective hull

- in general:

Proposition

Let $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{STopGr}$ *and* $\mathbf{B}$ *be a subcategory of* $\mathbf{A}$*. Moreover, let*

1. $\mathbf{B}_0 = \mathbf{B}$,
2. $\mathbf{B}_{\alpha+1} = \mathrm{MCH}_{\mathbf{A}}(\mathrm{S}\mathbf{B}_\alpha)$ *for every ordinal* α,
3. $\mathbf{B}_\beta = \bigcup_{\alpha<\beta} \mathbf{B}_\alpha$ *for every limit ordinal* β.

Then the hereditary coreflective hull of $\mathbf{B}$ *in* $\mathbf{A}$ *is the subcategory* $\mathbf{B}^* = \bigcup_{\alpha \in \mathrm{On}} \mathbf{B}_\alpha$.

The hereditary coreflective hull

- if extremal epimorphisms in **A** are precisely the surjective open homomorphisms:

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1. $\mathbf{B}_0 = \mathbf{B}$,
2. $\mathbf{B}_1$ *be the subcategory consisting of all coproducts of groups from* $\mathbf{B}_0$,
3. $\mathbf{B}_2$ *be the subcategory consisting of all subgroups of groups from* $\mathbf{B}_1$,
4. $\mathbf{B}_3$ *be the subcategory consisting of all extremal quotients of groups from* $\mathbf{B}_2$.

Then the hereditary coreflective hull of **B** *in* **A** *is the subcategory* $\mathbf{B}_3$.

The hereditary coreflective hull in $\mathbf{A} \subseteq \mathbf{STopAb}$

- if $\mathbf{A} \subseteq \mathbf{STopAb}$ is closed under the formation extremal quotients:

Proposition

Let $\mathbf{A}$ *be an epireflective subcategory of* $\mathbf{STopAb}$ *that is closed under the formation of extremal quotients and* $\mathbf{B}$ *be a coreflective subcategory of* $\mathbf{A}$*. Then the hereditary coreflective hull of* $\mathbf{B}$ *in* $\mathbf{A}$ *is the subcategory consisting of all subgroups of groups from* $\mathbf{B}$*.*

Hereditary bicoreflective subcategories

Proposition

Let **A** *be an extremal epireflective subcategory of* **STopGr** *or* **QTopGr** *and* **B** *be a hereditary coreflective subcategory of* **A** *that contains the group* Z *of integers with a* T_0*-topology. Then* **B** *contains the discrete group* $\mathbb{Z}$*, therefore it is bicoreflective in* **A**.

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Outline of proof:

- **B** is closed under the formation of finite products with the cross topology, since they are quotients of finite coproducts

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- $\mathbf{B}$ contains the group Z' of integers with a T_1-topology
- the subset $V = U \times U \setminus \{(n, n) : n \text{ is a non-zero integer}\}$ of $Z' \times^* Z'$ is open (U is a neighborhood of 0 in Z')

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! the proof fails in **PTopGr** and **TopGr**, since $Z \times^* Z$ does not need to be a paratopological group

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1. **A** *is closed under the formation of finite coproducts,*
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and **B** *be a hereditary coreflective subcategory of* **A** *that contains a group with a proper open subgroup. Then* **B** *contains the discrete group* $\mathbb{Z}$*, therefore it is bicoreflective in* **A**.

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Corollary

If **A** *satisfies the conditions of the preceding proposition,* **B** *is a hereditary coreflective subcategory of* **A** *that contains a cyclic group with a non-indiscrete topology that is not* T_0*, then* **B** *is bicoreflective in* **A**.

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Corollary

Let **A** *be an extremal epireflective subcategory of* **STopGr** *or* **QTopGr** *that satisfies the conditions of the preceding proposition. Then every hereditary coreflective subcategory of* **A** *that contains a non-indiscrete group is bicoreflective in* **A**.

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- hereditary coreflective subcategories that are not bicoreflective:
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  - is the hereditary coreflective hull of Z bicoreflective in **PTopGr** (**TopGr**)?

Hereditary bicoreflective subcategories in $\mathbf{A} \subseteq \mathbf{STopAb}$

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Example

- $\mathbf{A}$ – extremal epireflective in $\mathbf{PTopAb}$ or $\mathbf{TopAb}$, $\mathbb{Z} \in \mathbf{A}$
- $\mathbf{B}$ – such groups $G \in \mathbf{A}$ that every infinite cyclic subgroup of G that is T_0 has a neighborhood base at 0 consisting only of its non-trivial subgroups

$\Rightarrow$ $\mathbf{B}$ is hereditary and coreflective in $\mathbf{A}$, but not bicoreflective

If $r_{\mathbf{A}}(\mathbb{Z}) = \mathbb{Z}_n$

Proposition

Let $n \in \mathbb{N}$ and $\mathbf{A}$ be the subcategory of **STopGr** *(***QTopGr***,* **PTopGr** *or* **TopGr***) consisting of all groups G such that the order of every element of G is a divisor of n. Then every hereditary coreflective subcategory of $\mathbf{A}$ that contains a non-indiscrete group is bicoreflective in $\mathbf{A}$.*

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Proposition

*Let $\mathbf{A}$ be an epireflective subcategory of **STopAb** such that $r_{\mathbf{A}}(\mathbb{Z}) = \mathbb{Z}_n$, $n = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ be the prime factorization of n and $\mathbf{B}_i$ be the subcategory of $\mathbf{A}$ consisting precisely of such groups $G \in \mathbf{A}$ that no cyclic subgroup of G of order $p_i^{\alpha_i}$ is discrete. Then each $\mathbf{B}_i$ is a maximal hereditary coreflective subcategory of $\mathbf{A}$ that is not bicoreflective in $\mathbf{A}$.*

Thank you for your attention.