Characterizing Noetherian spaces as a Δ^0_2-analogue to compact spaces[1]

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Defining Noetherian spaces

Definition

A topological space $\mathbf{X}$ is called *Noetherian*, iff every strictly ascending chain of open sets is finite.

Theorem (Goubault-Larrecq)

The following are equivalent for a topological space $\mathbf{X}$*:*

1. $\mathbf{X}$ *is Noetherian, i.e. every strictly ascending chain of open sets is finite.*
2. *Every strictly descending chain of closed sets is finite.*
3. *Every open set is compact.*
4. *Every subset is compact.*

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Noetherian spaces occur as

- spectra of Noetherian rings.
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Quasi-Polish spaces

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A countably-based space is quasi-Polish, if its topology is induced by a Smyth-complete quasi-metric.

Proposition (de Brecht)

A locally compact sober countably-based space is quasi-Polish.

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When is a Noetherian space quasi-Polish?

Theorem

The following are equivalent for a sober Noetherian space $\mathbf{X}$*:*

1. $\mathbf{X}$ *is countable.*
2. $\mathbf{X}$ *is countably-based.*
3. $\mathbf{X}$ *is quasi-Polish.*

Baire Category Theorem in quasi-Polish spaces

Theorem (HECKMANN; BECHER & GRIGORIEFF)

Let $\mathbf{X}$ *be quasi-Polish. If* $\mathbf{X} = \bigcup_{i \in \mathbb{N}} A_i$ *with each* A_i *being* Σ^0_2*, then there is some* i_0 *such that* A_{i_0} *has non-empty interior.*

When is a quasi-Polish space Noetherian?

Theorem

The following are equivalent for a quasi-Polish space $\mathbf{X}$*:*

1. $\mathbf{X}$ *is Noetherian.*
2. *Every* Δ^0_2*-cover of* $\mathbf{X}$ *has a finite subcover.*

Corollary

A Noetherian quasi-Polish space is T_D *iff it is finite.*

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Represented spaces and computability

Definition

A *represented space* $\mathbf{X}$ is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ a surjective partial function.

Definition

$F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a realizer of $f : \mathbf{X} \to \mathbf{Y}$, iff $\delta_Y(F(p)) = f(\delta_X(p))$ for all $p \in \delta_X^{-1}(\mathrm{dom}(f))$. Abbreviate: $F \vdash f$.

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$f : \mathbf{X} \to \mathbf{Y}$ is called continuous, iff it has a continuous realizer.

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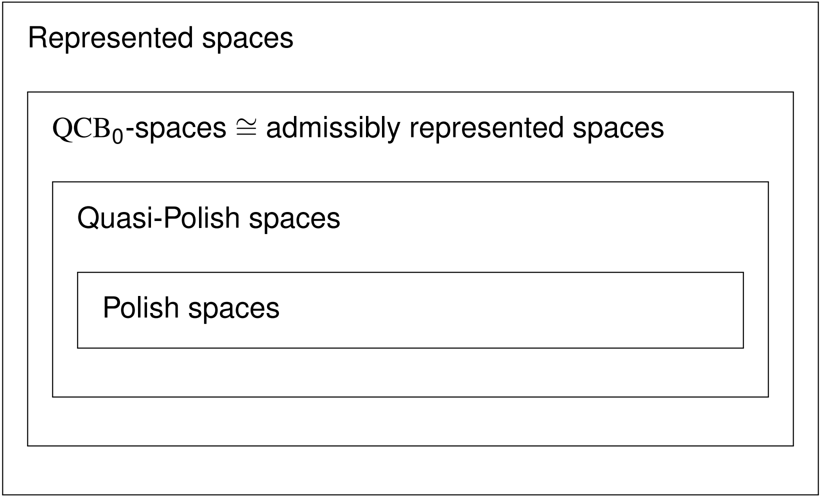
The various classes of spaces



Cartesian closure

Observation

We can form function spaces (to be denoted by $\mathcal{C}(-,-)$) in the category of represented spaces by the UTM-theorem/

Definition

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Definition

The space $\mathcal{O}(\mathbf{X})$ of open subsets of $\mathbf{X}$ is obtained from $\mathcal{C}(\mathbf{X}, \mathbb{S})$ via identification.

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Compactness in synthetic topology

Definition

Call a represented space $\mathbf{X}$ compact, if $\text{isFull} : \mathcal{O}(\mathbf{X}) \to \mathbb{S}$ is continuous.

Theorem

The following are equivalent for a represented space $\mathbf{X}$*:*

1. $\mathbf{X}$ *is compact.*
2. *For any represented space* $\mathbf{Y}$*, the map* $\forall : \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \to \mathcal{O}(\mathbf{Y})$ *mapping* R *to* $\{y \in \mathbf{Y} \mid \forall x \in \mathbf{X}\ (x, y) \in R\}$ *is continuous.*

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Δ^0_2-truth values

Definition

Let the represented space $\mathbb{S}^\nabla$ have the points $\{\top, \bot\}$ and the representation $\rho(w0^\omega) = \bot$ and $\rho(w1^\omega) = \top$.

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We can represent the Δ^0_2-subsets of **X** via their continuous characteristic functions $\mathcal{C}(\mathbf{X}, \mathbb{S}^\nabla)$.

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The main result

Theorem

A quasi-Polish space is Noetherian iff it is ∇-compact.

Definition

Let $\mathfrak{C}(\mathbf{X})$ denote the space of constructible subsets of $\mathbf{X}$.

Lemma

Let $\mathbf{X}$ be a Noetherian Quasi-Polish space. Then $\mathrm{id} : \Delta^0_2(\mathbf{X}) \to \mathfrak{C}(\mathbf{X})^\nabla$ *is well-defined and continuous.*

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