The Samuel realcompactification

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Introduction.

In this talk we will introduce a realcompactification for the class of uniform spaces (X, μ) and we will call it the **Samuel realcompactification**. Then we will study this realcompactification in the frame of metric spaces (X, d).

First, we will compare the Samuel realcompactification of a metric space (X, d) with another type of realcompactification that can be defined for metric spaces and which is called **Lipschitz realcompactification**.

Finally, we will see which metric spaces can be considered Samuel realcompact.

Contents.

1. General results about realcompactifications.
2. Examples of realcompactifications and compactifications.
3. Realcompactifications on metric spaces.
4. Equivalence of the Lipschitz and the Samuel realcompactification.
5. Samuel realcompact metric spaces.

Realcompactifications of a Tychonoff space.

Definition. A realcompactification of a Tychonoff space X is a realcompact space Y in which X is densely embedded.

compactification $\Rightarrow$ **realcompactification**

Order in the realcompactifications.

$(\mathfrak{R}(X), \leq)$ where $\mathfrak{R}(X) = \{$ realcompactifications of $X\}$ and $\leq$ is a partial order defined as follows:

$$\alpha_1 X \leq \alpha_2 X \text{ if there is } h : \alpha_2 X \to \alpha_1 X \text{ continuous,}$$

$$\text{leaving } X \text{ pointwise fixed}$$

Definition. Two realcompactifications $\alpha_1 X$ and $\alpha_2 X$ are equivalent whenever $\alpha_1 X \leq \alpha_2 X$ and $\alpha_2 X \leq \alpha_1 X$.

$$\exists\ h : \alpha_2 X \to \alpha_1 X \text{ an homeomorphism, leaving } X \text{ pointwise fixed}$$

Generation of realcompactifications.

$\mathcal{F} \subset C(X)$ separating points from closed sets

$$e : X \to \mathbb{R}^{\mathcal{F}} \text{ embedding}$$

$$e(x) = (f(x))_{f \in \mathcal{F}}$$

$$H(\mathcal{F}) = \overline{e(X)}^{\mathbb{R}^{\mathcal{F}}}$$

• $H(\mathcal{F})$ is the smallest realcompactification of X such that every function $f \in \mathcal{F}$ can be continuously extended to it.

• Whenever $\mathcal{F}$ has an algebraic structure, for instance, if $\mathcal{F}$ is a vector lattice, then

$H(\mathcal{F}) = \{$real unital vector lattice homomorphisms on $\mathcal{F}\}$

Compactifications.

$\mathcal{F}^* = \mathcal{F} \cap C^*(X)$ bounded functions of $\mathcal{F}$

• $H(\mathcal{F}^*)$ is the smallest compactification and realcompactification of X such that every function $f \in \mathcal{F}^*$ can be continuously extended to it.

$$X \subset H(\mathcal{F}) \subset H(\mathcal{F}^*)$$

Hewitt realcompactification and Stone-Čech compactification.

X Tychonoff space

$\mathcal{F} = C(X)$ real-valued continuous functions

$H(C(X)) = \upsilon X$ is the **Hewitt realcompactification** of X

• υX is largest element in the ordered family $(\mathfrak{R}(X), \leq)$.

• υX is the smallest realcompactification of X such that every $f \in C(X)$ is continuously extended.

Hewitt realcompactification and Stone-Čech compactification.

$\mathcal{F}^* = C^*(X)$ bounded real-valued continuous functions

$H(C^*(X)) = \beta X$ is the **Stone-Čech compactification of** X

• βX is the smallest compactification and realcompactification of X such that every $f \in C^*(X)$ is continuously extended.

$$X \subset \upsilon X \subset \beta X$$

Theorem. A Tychonoff space X is **realcompact** if and only if $X = \upsilon X$.

Samuel realcompactification and compactification.

(X, μ) uniform space

$\mathcal{F} = U_\mu(X)$ real-valued uniformly continuous functions

• $H(U_\mu(X))$ is the smallest realcompactification of X such that $f \in U_\mu(X)$ is continuously extended.

$\mathcal{F}^* = U^*_\mu(X)$ bounded real-valued uniformly continuous functions

$H(U^*_\mu(X)) = s_\mu X$ is the **Samuel compactification** of (X, μ)

• $s_\mu X$ is the smallest compactification and realcompactification of X such that every $f \in U^*_\mu(X)$ is continuously extended.

Samuel realcompactification and compactification.

$$X \subset H(U_\mu(X)) \subset s_\mu X$$

We will call $H(U_\mu(X))$ the **Samuel realcompactification** of (X, μ) because it is associated to the family of all the real-valued uniformly continuous functions as the Samuel compactification is associated to the family of all the bounded real-valued uniformly continuous functions.

Definition. A uniform space (X, μ) is **Samuel realcompact** if $X = H(U_\mu(X))$.

Samuel realcompact⇒realcompact

(X, d) metric space

$\mathcal{F} = Lip_d(X)$ real-valued Lipschitz functions

- $H(Lip_d(X))$ is the smallest realcompactification of X such that every $f \in Lip_d(X)$ is continuously extended.

We will call $H(Lip_d(X))$ the **Lipschitz realcompactification** of (X, d)

$\mathcal{F}^* = Lip_d^*(X)$ bounded real-valued Lipschitz functions

Theorem. $H(Lip_d^*(X))$ is exactly the Samuel compactification $s_d X$ of (X, d)

However in the unbounded case, $H(Lip_d(X))$ and $H(U_d(X))$ are in general different realcompactifications.

Lipschitz realcompactification.

$$X \subset H(Lip_d(X)) \subset s_d(X)$$

Definition. A metric space (X, d) is **Lipschitz realcompact** if $X = H(Lip_d(X))$.

Lipschitz realcompact $\Rightarrow$ **realcompact**

Main reference.

M. I. Garrido, A S. Meroño,
The Samuel realcompactification of a metric space
(submitted)

Lipschitz realcompactification.

Theorem. Let (X, d) be a metric space x_0 a fixed point in X and $B_d[x_0, n]$ the closed ball of center x_0 and radius $n \in \mathbb{N}$. Then

$$H(Lip_d(X)) = \bigcup_{n \in \mathbb{N}} cl_{s_d X} B_d[x_0, n] \subset s_d(X)$$

Corollary. A metric space is Lipschitz realcompact if and only if every closed bounded subset is compact.

Relations between the realcompactifications.

$$X \subset \upsilon X \subset H(U_d(X)) \subset H(Lip_d(X)) \subset s_d X$$

Lipschitz realcompact ⇒ Samuel realcompact

Observe that, uniformly equivalents metrics $\rho \overset{u}{\simeq} d$ define identical Samuel realcompactifications and compactifications.

$$H(U_d(X)) = \bigvee \{H(Lip_\rho(X)) : \rho \overset{u}{\simeq} d\} = \bigcap \{H(Lip_\rho(X)) : \rho \overset{u}{\simeq} d\}.$$

Relations between the realcompactifications.

We write $\rho \overset{t}{\simeq} d$ for topologically equivalent metrics.

$$\upsilon X = \bigvee \left\{ H(U_\rho(X)) : \rho \overset{t}{\simeq} d \right\} = \bigcap \left\{ H(U_\rho(X)) : \rho \overset{t}{\simeq} d \right\}$$

$$\upsilon X = \bigvee \left\{ H(Lip_\rho(X)) : \rho \overset{t}{\simeq} d \right\} = \bigcap \left\{ H(Lip_\rho(X)) : \rho \overset{t}{\simeq} d \right\}.$$

Problems.

1. To characterize those metric spaces (X, d) for which there exists a uniformly equivalent metric ρ such that $H(U_d(X))$ and $H(Lip_\rho(X))$ are equivalent realcompactifications, that is, $H(U_d(X)) = H(Lip_\rho(X))$.
2. To characterize those metric space (X, d) which are Samuel realcompact, that is, $X = H(U_d(X))$.

Bourbaki-bounded subsets.

$B_d(x, \varepsilon)$ be the open ball of center $x \in X$ and radius $\varepsilon > 0$

$$B_d^2(x, \varepsilon) = \bigcup \{B_d(y, \varepsilon) : y \in B_d(x, \varepsilon)\} \text{ and}$$

$$B_d^m(x, \varepsilon) = \bigcup \{B_d(y, \varepsilon) : y \in B_d^{m-1}(x, \varepsilon)\} \text{ whenever } m \geq 3.$$

Definition. A subset B of a metric space is Bourbaki-bounded if for every $\varepsilon > 0$ there exist finitely many points $x_1, \ldots, x_k \in X$ such that for some $m \in \mathbb{N}$,

$$B \subset \bigcup_{i=1}^{k} B_d^m(x_i, \varepsilon).$$

Bourbaki-bounded subsets.

Theorem. (ATSUJI) For a subset B of a metric space (X, d) the following statements are equivalent:

1. B is Bourbaki-bounded;
2. $f(B) \subset \mathbb{R}$ is bounded for every $f \in U_d(X)$.

Examples.

1. Totally bounded subsets of metric spaces are Bourbaki-bounded.
2. Bounded subsets of normed vector spaces are Bourbaki-bounded.
3. Let

$$X = \mathbb{N} \times \ell_2$$

where $\mathbb{N}$ brings the $0-1$ discrete metric and ℓ_2 is the classical Hilbert spaces. Let d the product metric. Then, in (X, d), bounded subset are not Bourbaki-bounded and Bourbaki-bounded subsets are not totally bounded.

Hejcman's problem.

$\mathbf{BB}_d(X) = \{$Bourbaki-bounded subsets $\}$

$\mathbf{B}_d(X) = \{$ bounded subsets $\}$

J. HEJCMAN,
On simple recognizing of bounded sets
Comment. Math. Univ. Carolinae, 38 (1997), 149-156.

To determine those metric spaces (X, d) such that for some uniformly equivalent $\rho \overset{u}{\simeq} d$, $\mathbf{BB}_d(X) = \mathbf{B}_\rho(X)$.

Example. Every normed vector space satisfies that $\mathbf{BB}_d(X) = \mathbf{B}_d(X)$.

Main result.

Proposition

For a metric space (X, d) the following statements are equivalent:

1. *there exists a uniformly equivalent metric $\rho \overset{u}{\simeq} d$ such that $H(U_d(X))$ is equivalent to $H(Lip_\rho(X))$;*
2. *$H(U_d(X))$ uniformly locally compact for the weak uniformity on $H(U_d(X))$ as a uniform subspace of the product space $\mathbb{R}^{U_d(X)}$;*
3. *every uniform partition of X is countable and there exists $\epsilon > 0$ such that for every $x \in X$ and every $m \in \mathbb{N}$, $B_d^m(x, \varepsilon)$ is a Bourbaki-bounded subset;*
4. *there exists a uniformly equivalent metric $\rho \overset{u}{\simeq} d$ such that $\mathbf{BB}_d(X) = \mathbf{B}_\rho(X)$.*

M. Hušek, A. Pulgarín,
Banach-Stone-Like theorems for lattices of uniformly continuous functions
Quest. Math. 35 (2012) 417-430.

M. I. Garrido, A S. Meroño,
The Samuel realcompactification of a metric space
(submitted)

Samuel realcompact metric spaces.

Proposition

A metric space (X, d) is Samuel realcompact, that is, $X = H(U_d(X))$, if and only if every uniform partition of X has non-measurable cardinal and X is Bourbaki-complete.

M. I. Garrido, A. S. Meroño,
New types of completeness in metric spaces
Ann. Acad. Sci. Fenn. Math. 39 (2014), 733-758.

Definition. A metric space is **Bourbaki-complete** if and only if every closed Bourbaki-bounded subset is compact.

Examples.

1. Every finite dimensional-Banach space is Samuel and Lipschitz realcompact because every closed and bounded set is compact.
2. Every infinite dimensional Banach space is not Samuel realcompact and not Lipschitz realcompact. In fact, unit closed ball is Bourbaki-bounded but not compact.
3. Every uniformly discrete metric space of non-measurable uncountable cardinality is Samuel realcompact but not Lipschitz realcompact.

To be Samuel realcompact is a stronger topological property than to be just realcompact. That is, not every metrizable realcompact space X is metrizable by a metric d such that (X, d) is Samuel realcompact.

A. Hohti, H. Junnila and A. S Meroño
On Strongly Čech-complete spaces
(manuscript)

Theorem. A metrizable space is metrizable by a Bourbaki-complete metric if and only if it is homeomorphic to a closed subspace of $\mathbb{R}^\omega \times \kappa^\omega$ where κ^ω is the Baire space of weight κ.

Corollary. A metrizable space is metrizable by a metric d such that (X, d) is Samuel realcompact if and only if it is homeomorphic to a closed subspace of $\mathbb{R}^\omega \times \kappa^\omega$ where κ^ω is the Baire space of weight κ and κ is a non-measurable cardinal.

Corollary. A connected metrizable space is metrizable by a metric d such that (X, d) is Samuel realcompact if and only if it is Čech-complete and separable.

THANK YOU VERY MUCH!