Ideal convergence of nets of functions with values in uniform spaces

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In recent years, a lot of papers have been written on statistical convergence and ideal convergence in metric and topological spaces (see, for instance, [14, 15, 17, 18, 19, 20, 22, 23]).

Recently, several researchers have been working on sequences of real functions and of functions between metric spaces by using the idea of statistical and $\mathcal{I}$-convergence (see, for instance, [2, 3, 6, 7, 8, 9]).

On the other hand, classical results about sequences and nets of functions have been extended from metric to uniform spaces (see, for example, [5, 16, 21]).

In this talk, we investigate the pointwise, uniform, quasi-uniform, and the almost uniform $\mathcal{I}$-convergence for a net $(f_d)_{d\in D}$ of functions of an arbitrary topological space X into a uniform space Y, where $\mathcal{I}$ is an ideal on D. Particularly, the continuity of the limit of the net $(f_d)_{d\in D}$ is studied. Since each metric space is a uniform space, the results remain valid in the case that Y is a metric space.

The rest of the talk is organized as follows. Section 1 contains preliminaries. In section 2 we give the pointwise, uniform and quasi-uniform $\mathcal{I}$-convergence for nets of functions with values in uniform spaces. In section 3 we present a modification of the classical result which states that equicontinuity on a compact metric space turns pointwise to uniform convergence. In section 4 we extend the classical result of Arzelà [1] to the quasi uniform $\mathcal{I}$-convergence of nets of functions with values in uniform spaces. Finally, the concept of almost uniform $\mathcal{I}$-convergence of a net of function with values in a uniform space is investigated in sections 5 and 6.

Outline

1. Preliminaries

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Uniformity

A uniformity on a set Y is a collection $\mathcal{U}$ of subsets of $Y \times Y$ satisfying the following properties:

($\mathcal{U}_1$) $\Delta \subseteq U$, for every $U \in \mathcal{U}$, where $\Delta = \{(y, y) : y \in Y\}$.

($\mathcal{U}_2$) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$, where $U^{-1} = \{(y_1, y_2) : (y_2, y_1) \in U\}$.

($\mathcal{U}_3$) If $U \in \mathcal{U}$ and $U \subseteq V \subseteq Y \times Y$, then $V \in \mathcal{U}$.

($\mathcal{U}_4$) If $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$.

($\mathcal{U}_5$) For every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$, where $V \circ V = \{(y_1, y_2) : \exists\, y \in Y \text{ such that } (y_1, y) \in V \text{ and } (y, y_2) \in V\}$.

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Uniform space

A uniform space is a pair $(Y, \mathcal{U})$ consisting of a set Y and a uniformity $\mathcal{U}$ on the set Y. The elements of $\mathcal{U}$ are called entourages. An entourage V is called symmetric if $V^{-1} = V$. For every $U \in \mathcal{U}$ and $y_0 \in Y$ we use the following notation:

$$U[y_0] = \{y \in Y : (y_0, y) \in U\}.$$

Lemma

Let $(Y, \mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then, there exists a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$.

Uniform topology

For every uniform space $(Y, \mathcal{U})$ the uniform topology $\tau_{\mathcal{U}}$ on Y is family consisting of the empty set and all subsets O of Y such that for each $y \in O$ there is $U \in \mathcal{U}$ with $U[y] \subseteq O$.

If (Y, ρ) is a metric space, then the collection $\mathcal{U}_\rho$ of all $U \subseteq Y \times Y$ for which there is $\varepsilon > 0$ such that

$$\{(y_1, y_2) : \rho(y_1, y_2) < \varepsilon\} \subseteq U$$

is a uniformity on Y which generates a uniform space with the same topology as the topology induced by ρ.

For the special case in which $Y = [0, 1]$ and $\rho(y_1, y_2) = |y_1 - y_2|$, then we call $\mathcal{U}_\rho$ the usual uniformity for $[0, 1]$.

Lemma

Let $(X,\mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then, there exists a symmetric entourage $W \in \mathcal{U}$ such that:

1. $W \subseteq U$.
2. W is open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$.

Lemma

Let $(X,\mathcal{U})$ be a uniform space and $U \in \mathcal{U}$. Then, there exists a symmetric entourage $K \in \mathcal{U}$ such that:

1. $K \subseteq U$.
2. K is closed in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$.

Continuous mapping

A mapping f of a topological space X into a uniform space $(Y, \mathcal{U})$ is called continuous at x_0 if for each $U \in \mathcal{U}$ there exists an open neighbourhood O_{x_0} of x_0 such that

$$f(O_{x_0}) \subseteq U[f(x_0)]$$

or equivalently

$$(f(x_0), f(x)) \in U, \text{ for every } x \in O_{x_0}.$$

The mapping f is called continuous if it is continuous at every point of X.

Ideal

Let D be a nonempty set. A family $\mathcal{I}$ of subsets of D is called an ideal on D if $\mathcal{I}$ has the following properties:

1. $\emptyset \in \mathcal{I}$.
2. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
3. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

Non-trivia Ideal

An ideal $\mathcal{I}$ on D is said to be non-trivial if $\mathcal{I} \neq \{\emptyset\}$ and $D \notin \mathcal{I}$. The ideal $\mathcal{I}$ is called admissible if it contains all finite subsets of D.

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Directed set

A partially ordered set D is called directed if every two elements of D have an upper bound in D.

Let $(D, \leqslant)$ be a directed set. We consider the family

$$\{A \subseteq D : A \subseteq \{d \in D : d \not\geqslant d_0\} \text{ for some } d_0 \in D\}.$$

This family is an ideal on D which will be denoted by $\mathcal{I}_D$.

Net

A net in the set Y^X of all functions $f : X \to Y$ is an arbitrary function s from a nonempty directed set D to Y^X. If $s(d) = f_d$, for all $d \in D$, then the net s will be denoted by the symbol $(f_d)_{d \in D}$.

Semi-subnet

If $(f_d)_{d\in D}$ is a net in Y^X, then a net $(g_\lambda)_{\lambda\in\Lambda}$ in Y^X is said to be a semi-subnet of $(f_d)_{d\in D}$ if there exists a function $\varphi : \Lambda \to D$ such that $g_\lambda = f_{\varphi(\lambda)}$, for every $\lambda \in \Lambda$. We write $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ to indicate the fact that φ is the function mentioned above.

Suppose that $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ is a semi-subnet of the net $(f_d)_{d\in D}$. For every ideal $\mathcal{I}$ of the directed set D, we consider the family

$$\{A \subseteq \Lambda : \varphi(A) \in \mathcal{I}\}.$$

This family is an ideal on Λ which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

$\mathcal{I}$-convergence

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions of a nonempty set X into a metric space (Y,ρ), and let $\mathcal{I}$ be an ideal on D.

1. $(f_n)_{n\in\mathbb{N}}$ is said to $\mathcal{I}$-pointwise converge to f on X if for every $x \in X$ and for every $\varepsilon > 0$ there exists $A \in \mathcal{I}$ such that for every $n \notin A$ we have $\rho(f(x), f_n(x)) < \varepsilon$.
2. $(f_n)_{n\in\mathbb{N}}$ is said to $\mathcal{I}$-uniform converge to f on X if for every $\varepsilon > 0$ there exists $A \in \mathcal{I}$ such that for every $x \in X$ and for every $n \notin A$ we have $\rho(f(x), f_n(x)) < \varepsilon$.

Quasi uniform convergence

A net $(f_d)_{d\in D}$ of functions of a nonempty set X into a metric space (Y,ρ) is said to converge quasi uniformly to f on X if it converges pointwise to f, and for every $\varepsilon > 0$ and for every $d_0 \in D$, there exists a finite number of indices $d_1,\dots,d_k \geqslant d_0$ such that for each $x \in X$ at least one of the following inequalities holds:

$$\rho(f(x), f_{d_i}(x)) < \varepsilon,\ i = 1,\dots,k.$$

Almost uniform convergence

A net $(f_d)_{d\in D}$ of functions of a nonempty set X into a metric space (Y, ρ) is said to converge almost uniformly to f on X if for every $x \in X$, for every $\varepsilon > 0$, and for every $d \in D$, there exist $d_x \geqslant d$ and an open neighbourhood O_x of x such that for every $t \in O_x$ we have $\rho(f(t), f_{d_x}(t)) < \varepsilon$.

Completely regular space

A topological space X is called completely regular if X is a T_1-space and for every closed subset F of X and for every point $x \in X \setminus F$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

Locally compact space

A topological space X (not necessarily Hausdorff) is called locally compact if for each $x \in X$ there exist an open neighbourhood U of x and a compact subset C of X such that $U \subseteq C$.

Pseudocompact space

A topological space X (not necessarily completely regular) is called pseudocompact if every continuous real-valued function on X is bounded.

A completely regular space X is pseudocompact if and only if every locally finite collection of nonempty open subsets of X is finite.

In this section we consider a net $(f_d)_{d\in D}$ of functions of a topological space X into a uniform space $(Y,\mathcal{U})$, and an ideal $\mathcal{I}$ on D.

Pointwise $\mathcal{I}$-convergence

The net $(f_d)_{d\in D}$ is said to $\mathcal{I}$-converge pointwise to f on X if for every $x\in X$ and for every $U\in\mathcal{U}$ there exists $A\in\mathcal{I}$ such that for every $d\notin A$ we have $(f(x),f_d(x))\in U$. In this case we write $(f_d)_{d\in D}\xrightarrow{\mathcal{I}} f$. We shall say that the net $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise on X if there is a function to which the net $\mathcal{I}$-converges pointwise.

Proposition 2.1

If $(f_d)_{d\in D}\xrightarrow{\mathcal{I}} f$, then for every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ we have $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}\xrightarrow{\mathcal{I}_\Lambda(\varphi)} f$.

Uniform $\mathcal{I}$-convergence

The net $(f_d)_{d\in D}$ is said to $\mathcal{I}$-converge uniformly to f on X if for every $U \in \mathcal{U}$ there exists $A \in \mathcal{I}$ such that for every $x \in X$ and for every $d \notin A$ we have $(f(x), f_d(x)) \in U$. In this case we write $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$. We shall say that the net $(f_d)_{d\in D}$ $\mathcal{I}$-converges uniformly on X if there is a function to which the net $\mathcal{I}$-converges uniformly.

Proposition 2.2

If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$, then for every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ we have $(g_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi)\text{-}u} f$.

Proposition 2.3

If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$, the functions f_d, $d \in D$ are continuous, and the ideal $\mathcal{I}$ is non-trivial, then the function f is continuous.

Quasi-uniform $\mathcal{I}$-convergence

The net $(f_d)_{d\in D}$ is said to $\mathcal{I}$-converge quasi-uniformly to f on X if $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$ and for every $U \in \mathcal{U}$ and for every $A \in \mathcal{I} \setminus \{D\}$, there exists a finite subset $\{d_1, \ldots, d_n\}$ of $D \setminus A$ such that for each $x \in X$ at least one of the following relations holds:

$$(f(x), f_{d_i}(x)) \in U, \ i = 1, \ldots, n.$$

In this case we write $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}qu} f$. We shall say that the net $(f_d)_{d\in D}$ $\mathcal{I}$-converges quasi uniformly on X if there is a function to which the net $\mathcal{I}$-converges quasi-uniformly.

Proposition 2.4

If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$ and $(g_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi)\text{-}qu} f$ for some semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$, where $\mathcal{I}_\Lambda(\varphi)$ is a non-trivial ideal on Λ, then $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}qu} f$.

Equicontinuous family [16]

A family $\{f_i : i \in I\}$ of functions of a topological space X into a uniform space $(Y, \mathcal{U})$ is called equicontinuous at a point x_0 of X if for every $U \in \mathcal{U}$ there exists an open neighbourhood O_{x_0} of x_0 such that $(f_i(x_0), f_i(x)) \in U$ for all $i \in I$ and for all $x \in O_{x_0}$.

The family $\{f_i : i \in I\}$ is called equicontinuous if it is equicontinuous at each point of X.

$\mathcal{I}$-equicontinuous family

Let $(f_d)_{d \in D}$ be a net of functions of a topological space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D. The family $\{f_d : d \in D\}$ is called $\mathcal{I}$-equicontinuous at a point x_0 of X if for every $U \in \mathcal{U}$ there exist $A \in \mathcal{I}$ and an open neighbourhood O_{x_0} of x_0 such that $(f_d(x_0), f_d(x)) \in U$ for all $d \in D \setminus A$ and for all $x \in O_{x_0}$.

The family $\{f_d : d \in D\}$ is called $\mathcal{I}$-equicontinuous if it is equicontinuous at each point of X.

Theorem 3.1

Let $(f_d)_{d\in D}$ be a net of functions of a topological space X into a uniform space $(Y,\mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D such that the family $\{f_d : d \in D\}$ is $\mathcal{I}$-equicontinuous. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, then the function f is continuous. Moreover, the $\mathcal{I}$-convergence is uniform on every compact subset of X.

Corollary 3.1

Let $(f_d)_{d\in D}$ be a net of functions of a topological space X into a uniform space $(Y,\mathcal{U})$, where the family $\{f_d : d \in D\}$ is equicontinuous and let $\mathcal{I}$ be a non-trivial ideal on D. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, then the function f is continuous. Moreover, the $\mathcal{I}$-convergence is uniform on every compact subset of X.

Lemma 4.1

Let f and g be two continuous functions of a topological space X into a uniform space $(Y, \mathcal{U})$. The following statements are true:

1. The function $m : X \to (Y \times Y, \tau_{\mathcal{U}} \times \tau_{\mathcal{U}})$ defined by $m(x) = (f(x), g(x))$, for every $x \in X$ is continuous.
2. If W is open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$, then the set $\{x \in X : (f(x), g(x)) \in W\}$ is open.

Lemma 4.2

Let f be a continuous function of a topological space X into a uniform space $(Y, \mathcal{U})$ and let $x_0 \in X$.

1. The function $m : X \to (Y \times Y, \tau_{\mathcal{U}} \times \tau_{\mathcal{U}})$ defined by $m(x) = (f(x), f(x_0))$, for every $x \in X$ is continuous.
2. If W is open in the product topology $\tau_{\mathcal{U}} \times \tau_{\mathcal{U}}$ of $Y \times Y$, then the set $\{x \in X : (f(x_0), f(x)) \in W\}$ is open.

Theorem 4.1

Let $(f_d)_{d \in D}$ be a net of continuous functions of a topological space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D. If the net $(f_d)_{d \in D}$ $\mathcal{I}$-converges pointwise to a continuous limit, then the $\mathcal{I}$-convergence is quasi-uniform on every compact subset of X. Conversely, if the net $(f_d)_{d \in D}$ $\mathcal{I}$-converges quasi-uniformly on a subset of X, then the limit is continuous on this subset.

Corollary 4.1

On a compact topological space, the limit of a pointwise $\mathcal{I}$-convergent net $(f_d)_{d\in D}$ of continuous functions from a topological space into a uniform space is continuous if and only if the $\mathcal{I}$-convergence is quasi-uniform, when $\mathcal{I}$ is a non-trivial ideal on D.

Corollary 4.2

Let X be a compact topological space, and suppose that the net $(f_d)_{d\in D}$ of continuous functions of the topological space X into a uniform space $(Y,\mathcal{U})$ $\mathcal{I}$-converges pointwise to a continuous function f, where $\mathcal{I}$ is a non-trivial ideal on D. Then, f is continuous in any topology on X in which all the functions f_d, $d \in D$ are continuous.

Theorem 4.2

Let M be a dense subset of a compact topological space X, and suppose that the net $(f_d)_{d\in D}$ of continuous functions of X into the uniform space $(Y,\mathcal{U})$ $\mathcal{I}$-converges pointwise to a continuous limit f on M, where $\mathcal{I}$ is a non-trivial ideal on D. The following statements are true:

1. If $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to f on X, then every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ $\mathcal{I}_\Lambda(\varphi)$-converges quasi-uniformly to f on X, in the case where $\mathcal{I}_\Lambda(\varphi)$ is a non-trivial ideal on Λ.
2. If every semi-subnet $(g_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(f_d)_{d\in D}$ $\mathcal{I}_\Lambda(\varphi)$-converges quasi-uniformly to f on M, then $(f_d)_{d\in D}$ $\mathcal{I}$-converges pointwise to f on X.

Almost uniform $\mathcal{I}$-convergence

A net $(f_d)_{d\in D}$ of functions of a topological space X with values in a uniform space $(Y,\mathcal{U})$ is said to $\mathcal{I}$-converge almost uniformly to f on X if for every $x \in X$ and for every $U \in \mathcal{U}$ there exist $A \in \mathcal{I}$ and an open neighbourhood O_x of x such that for every $d \notin A$ and for every $t \in O_x$ we have $(f(t), f_d(t)) \in U$. In this case we write $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$. We shall say that the net $(f_d)_{d\in D}$ $\mathcal{I}$-converges almost uniformly on X if there is a function to which the net $\mathcal{I}$-converges almost uniformly.

Theorem 5.1

Let $(f_d)_{d\in D}$ be a net of continuous functions of a topological space X into a uniform space $(Y,\mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$, then the function f is continuous.

Theorem 5.2

Let $(f_d)_{d\in D}$ be a net of functions of a topological space X into a uniform space $(Y,\mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D such that the family $\{f_d : d \in D\}$ is $\mathcal{I}$-equicontinuous. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, where the function f is continuous, then the $\mathcal{I}$-convergence is almost uniform.

Corollary 5.1

Let $(f_d)_{d\in D}$ be a net of functions of a topological space X into a uniform space $(Y,\mathcal{U})$, where the family $\{f_d : d \in D\}$ is equicontinuous and let $\mathcal{I}$ be a non-trivial ideal on D. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$, where the function f is continuous, then the $\mathcal{I}$-convergence is almost uniform.

Proposition 6.1

Let $(f_d)_{d\in D}$ be a net of functions from a topological space X into a uniform space $(Y, \mathcal{U})$. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$, then $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$.

Theorem 6.1

Let $(f_d)_{d\in D}$ be a net of functions from a compact space X into a uniform space $(Y, \mathcal{U})$ and let $\mathcal{I}$ be a non-trivial ideal on D. If $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}au} f$, then $(f_d)_{d\in D} \xrightarrow{\mathcal{I}\text{-}u} f$.

Example 6.1

Let X be a completely regular non-pseudocompact space. Since X is not pseudocompact, there exists a locally finite family $\mathcal{F}$ of nonempty open sets which is not finite. Let $\preceq$ be a well-order in $\mathcal{F}$ and let α be the order type of $(\mathcal{F}, \preceq)$. By D we denote the directed set of all ordinal numbers less than α. Hence, the family $\mathcal{F}$ can be presented as $\{U_d : d \in D\}$. For each $d \in D$ we select a point $x_d \in U_d$. Since X is completely regular, there exists a continuous function $f_d : X \to [0, 1]$ such that $f_d(x_d) = 0$ and $f_d(X \setminus U_d) = \{1\}$.

Consider the function $f : X \to [0, 1]$ defined by $f(t) = 1$, for every $t \in X$.

Let $\mathcal{I}$ be an admissible non-trivial ideal on D and $\mathcal{U}$ be the usual uniformity for $[0, 1]$.

- $(f_d)_{d \in D} \xrightarrow{\mathcal{I}\text{-}au} f$.
- $(f_d)_{d \in D}$ does not $\mathcal{I}$-converge uniformly to f.

Example 6.2

Let X be a completely regular space which is not locally compact. Since X is not locally compact, there exists $x \in X$ such that for each open neighbourhood O of x and for each compact set C we have $O \nsubseteq C$. Let $\mathcal{O}(x)$ be the family of all open neighbourhoods of x and let $\mathcal{C}$ be the family of all nonempty compact subsets of X. We consider the directed set $(D, \leqslant)$, where $D = \mathcal{O}(x) \times \mathcal{C}$ and

$$(O_1, C_1) \leqslant (O_2, C_2) \text{ if and only if } O_2 \subseteq O_1 \text{ and } C_1 \subseteq C_2.$$

For each $d = (O, C) \in D$ we select a point $x_d \in O \setminus C$. Since X is completely regular, there exists a continuous function $f_d : X \to [0, 1]$ such that $f_d(x_d) = 0$ and $f_d((X \setminus O) \cup C) = \{1\}$.

Example 6.2 (cont.)

Consider the function $f : X \to [0,1]$ defined by $f(t) = 1$, for every $t \in X$.

Let

$$\mathcal{I}_D = \{A \subseteq D : A \subseteq \{d \in D : d \not\geq d_0\} \text{ for some } d_0 \in D\}$$

and $\mathcal{U}$ be the usual uniformity for $[0,1]$.

- $(f_d)_{d \in D} \xrightarrow{\mathcal{I}_D\text{-}u} f$ on every compact subset of X.
- $(f_d)_{d \in D}$ does not $\mathcal{I}_D$-converge almost uniformly to f.

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