Resolvable-measurable mappings of metrizable spaces

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Definition

A subset E of a space X is *resolvable* if it can be represented in the following form:

$$E = (F_1 \setminus F_2) \cup (F_3 \setminus F_4) \cup \ldots \cup (F_\xi \setminus F_{\xi+1}) \cup \ldots,$$

where $\langle F_\xi \rangle$ forms a decreasing transfinite sequence of closed sets in X.

Notice that every resolvable subset of a metrizable space X is a Δ^0_2-set, i.e., a set that is both F_σ and G_δ in X.

Definitions

A mapping $f\colon X \to Y$ is said to be

- *resolvable-measurable* if $f^{-1}(U)$ is a resolvable subset of X for every open set $U \subset Y$;
- Δ^0_2-*measurable* if $f^{-1}(U) \in \Delta^0_2(X)$ for every open set $U \subset Y$;
- G_δ-*measurable* if $f^{-1}(U) \in G_\delta(X)$ for every open set $U \subset Y$;
- *countably continuous* if X has a countable cover $\mathcal{C}$ such that the restriction $f \restriction C$ is continuous for every $C \in \mathcal{C}$;
- *piecewise continuous* if X has a countable *closed* cover $\mathcal{C}$ such that the restriction $f \restriction C$ is continuous for every $C \in \mathcal{C}$.

Historical notes

Decomposition of a mapping $f\colon X \to Y$ into a countable sum of continuous mappings was studied in many works. The first significant result is the following

Theorem 1.[J.E. Jayne, C.A. Rogers (1982)]

Let $f\colon X \to Y$ be a mapping of an absolute Souslin-$\mathcal{F}$ set X to a metric space Y.
Then f is Δ^0_2-measurable if and only if it is piecewise continuous.

Kačena, Motto Ros, and Semmes (2012) showed that Theorem 1 holds for a regular space Y.

Historical notes

Theorem 2. [J. Pawlikowski, M. Sabok (2012)]

Let $f\colon X \to Y$ be a Borel function from an analytic space X to a separable metrizable space Y.
Then either f is countably continuous, or else there is topological embedding of the Pawlikowski function P into f.

Theorem 3. [A.V. Ostrovsky, 2016]

Let X and Y be separable zero-dimensional metrizable spaces.
Then every resolvable-measurable mapping $f\colon X \to Y$ is countably continuous.

The main result

1 **Theorem 4.**

Every resolvable-measurable mapping $f\colon X \to Y$ of a metrizable space X to a regular space Y is piecewise continuous.

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2 **Corollary 5.**

Let $f\colon X \to Y$ be a bijection between metrizable spaces X and Y such that f and f^{-1} are both resolvable-measurable mappings.
Then:
1) $\dim X = \dim Y$;
2) X is an absolute F_σ-set $\Leftrightarrow$ Y is an absolute F_σ-set.

Completely Baire space

Definition

A space X is *completely Baire* (or *hereditarily Baire*) if every closed subset of X is a Baire space.

Lemma 6.

For a metrizable space X the following conditions are equivalent:

(i) no closed subspace of X is homeomorphic to the space $\mathbb{Q}$ of rational numbers,

(ii) X is a completely Baire space,

(iii) the family of $\Delta_2^0(X)$-sets coincides with the family of resolvable sets in X.

Completely Baire space

Theorem 7.

Let $f\colon X \to Y$ be a mapping of a metrizable completely Baire space X to a regular space Y. Then the following conditions are equivalent:

(i) f is resolvable-measurable;

(ii) f is piecewise continuous;

(iii) f is G_δ-measurable.

Equivalence (ii) $\Leftrightarrow$ (iii) was obtained by T. Banakh and B. Bokalo (2010).

Completely Baire space

The following statement shows that in the study of F_σ-measurable mappings sometimes it suffices to consider separable spaces.

Theorem 8.

Let $f\colon X \to Y$ be an F_σ-measurable mapping of a metrizable completely Baire space X to a regular space Y. If the restriction $f \restriction Z$ is piecewise continuous for any zero-dimensional separable closed subset Z of X, then f is piecewise continuous.

References

1) J.E. Jayne and C.A. Rogers, *First level Borel functions and isomorphisms*, J. Math. pures et appl., 61 (1982), 177–205.
2) T. Banakh and B. Bokalo, *On scatteredly continuous maps between topological spaces*, Topol. Applic., 157 (2010), 108–122.
3) M. Kačena, L. Motto Ros, and B. Semmes, *Some observations on "A new proof of a theorem of Jayne and Rogers"*, Real Analysis Exchange, 38 (2012/2013), no. 1, 121–132.
4) A. Ostrovsky, *Luzin's topological problem*, preprint,2016.
5) J. Pawlikowski and M. Sabok, *Decomposing Borel functions and structure at finite levels of the Baire hierarchy*, Annals of Pure and Applied Logic, 163 (2012) 1784–1764.