Automorphisms of $\mathscr{P}(\lambda)/\mathscr{I}_\kappa$

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- We will be working with the Boolean algebras $\mathscr{P}(\lambda)/\mathscr{I}_\kappa$.
- $[A]$ will denote the equivalence class of a set $A \subseteq \lambda$.
- Every Question is open (as far as I know).

Question

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This problem was almost completely solved in the 1970's:

Theorem (Balcar-Frankiewicz, 1978)

Suppose $\kappa < \lambda$ and $\mathscr{P}(\kappa)/\,\mathrm{fin}$ and $\mathscr{P}(\lambda)/\,\mathrm{fin}$ are isomorphic. Then $\kappa = \omega$ and $\lambda = \omega_1$.

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Question (The Katowice Problem)

Is it consistent with ZFC that $\mathscr{P}(\omega)/\,\mathrm{fin}$ and $\mathscr{P}(\omega_1)/\,\mathrm{fin}$ are isomorphic?

Theorem (Chodounský, Dow, Hart, de Vries, 2015)

Suppose $\mathscr{P}(\omega)/\text{fin}$ *and* $\mathscr{P}(\omega_1)/\text{fin}$ *are isomorphic. Then there is a nontrivial automorphism of* $\mathscr{P}(\omega)/\text{fin}$*.*

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A natural question to ask is: what effect does an isomorphism $\mathscr{P}(\omega)/\text{fin} \simeq \mathscr{P}(\omega_1)/\text{fin}$ have on the automorphism group of $\mathscr{P}(\omega_1)/\text{fin}$?

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A natural question to ask is: what effect does an isomorphism $\mathscr{P}(\omega)/\operatorname{fin} \simeq \mathscr{P}(\omega_1)/\operatorname{fin}$ have on the automorphism group of $\mathscr{P}(\omega_1)/\operatorname{fin}$?

In particular, what automorphisms of $\mathscr{P}(\omega)/\operatorname{fin}$ have properties that are interesting when given to automorphisms of $\mathscr{P}(\omega_1)/\operatorname{fin}$?

For example: if σ is the shift automorphism of $\mathscr{P}(\omega)/$ fin, i.e.

$$\sigma([A]) = [\{n+1 \mid n \in A\}]$$

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then σ has no fixed points other than $[\emptyset]$ and $[\omega]$.

Question

Is it consistent with ZFC that there exists an automorphism of $\mathscr{P}(\omega_1)/\text{fin}$ whose only fixed points are $[\emptyset]$ and $[\omega_1]$?

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Definition

An automorphism π of $\mathscr{P}(\lambda)/\mathscr{I}$ is cardinality-preserving if for every $A \subseteq \lambda$ there is $B \subseteq \lambda$ with the same cardinality such that $\pi([A]) = [B]$.

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(An automorphism of $\mathscr{P}(\omega_1)/$ fin could take a countable set to an uncountable set, or vice-versa, if $\mathscr{P}(\omega)/\text{fin} \simeq \mathscr{P}(\omega_1)/\text{fin}$).

Theorem (M.-Larson)

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The proof goes through ladder systems.

Definition

A ladder system on a set S of ordinals is a collection $L_\alpha \subseteq \alpha$ ($\alpha \in S$) such that each L_α is cofinal in α and has order-type $\text{cf}(\alpha)$.

Definition

A ladder system L_α ($\alpha \in S$) has κ-uniformization if for every family of colorings $f_\alpha : L_\alpha \to \kappa$, there is a function $F : \sup(S) \to \kappa$ such that for all $\alpha \in S$, $F \restriction L_\alpha =^* f_\alpha$.

Theorem (M.-Larson)

Let π be a cardinality-preserving automorphism of $\mathscr{P}(\omega_1)/\mathrm{fin}$, and let

$$S_0 = \{\alpha < \omega_1 \mid \pi([\alpha]) \not\leq [\alpha]\}$$
$$S_1 = \{\alpha < \omega_1 \mid \pi([\alpha]) \not\geq [\alpha]\}$$

Then for each $i < 2$ there is a ladder system on a club subset of S_i which satisfies 2-uniformization.

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Theorem (Devlin-Shelah)

Suppose S_0 and S_1 are subsets of ω_1 such that $S_0 \cup S_1$ contains a club, and each S_i supports a ladder system with 2-uniformization. Then $2^\omega = 2^{\omega_1}$.

Proof (for S_0): Suppose $\pi([\alpha]) \not\leq [\alpha]$ but $\pi([\beta]) \leq [\alpha]$ for all $\beta < \alpha$. (This is where we need a club subset of S_0.)

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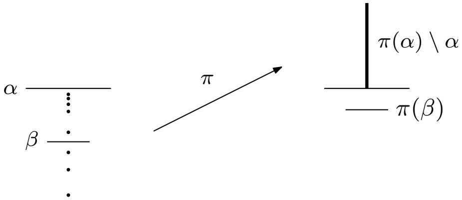

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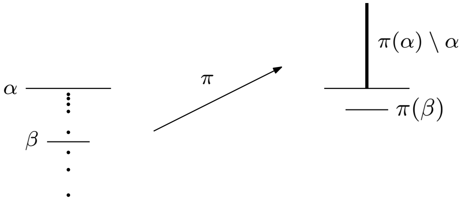
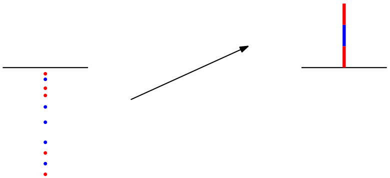


The *excess*, when pulled back through π^{-1}, is infinite and has finite intersection with every $\beta < \alpha$, hence is cofinal in α and has order-type ω. Call this set L_α.

Now suppose a coloring $f_\alpha : L_\alpha \to 2$ is given for each such α. Then π defines colorings on each excess set.



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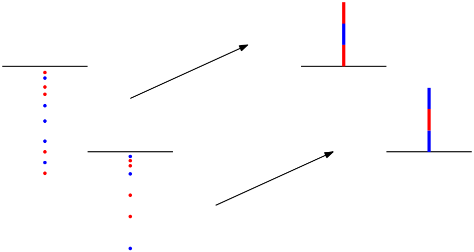

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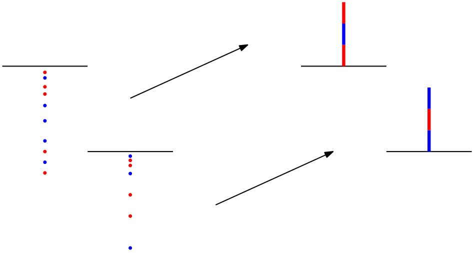

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Then we can just put their colorings together. The image of this coloring under π^{-1} uniformizes the f_α's. □

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Question

Is there consistently a nontrivial automorphism of $\mathscr{P}(\lambda)/$ ctble for some uncountable λ?

Definition

We say that a set of reals $X \subseteq \mathbb{R}$ is a Q_B-set if for every $Y \subseteq X$, there is a Borel $B \subseteq \mathbb{R}$ such that $B \cap X = Y$.

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Remark

If there is a Q_B-set X of size λ, then there is a countable family of subsets of λ which generates $\mathscr{P}(\lambda)$ as a σ-algebra. Note also that $\mathscr{P}(\lambda)/$ ctble is countably complete (assuming $\operatorname{cf} \lambda > \omega$).

Theorem (Larson-M.)

Suppose there exists a Q_B-set X of size λ, where $\operatorname{cf}\lambda > \omega$. Then the following are equivalent.

1. *There exists a Q_B set Y of size λ such that $X \cap Y = \emptyset$ and for every Borel B, $|B \cap X| + \aleph_0 = |B \cap Y| + \aleph_0$.*

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Corollary

Suppose that there exists a Q_B-set of size λ, and for every pair of Q_B-sets X and Y of size λ, $X \cup Y$ is also a Q_B-set. Then there is no nontrivial, cardinality-preserving automorphism of $\mathscr{P}(\lambda)/\operatorname{ctble}$.

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$$\pi : \mathscr{P}(X)/\,\text{ctble} \to \mathscr{P}(Y)/\,\text{ctble}$$

such that there is no bijection $f : X \to Y$ such that $\pi([A]) = [f''(A)]$ for all $A \subseteq X$.

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Here it is: for all Borel $B \subseteq \mathbb{R}$,

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- π is nontrivial: suppose $f : X \to Y$ is such that $f''(B \cap X) \triangle (B \cap Y)$ is countable for every Borel B. Then there is a countable set S such that

  $$f''(N \cap X) \triangle (N \cap Y) \subseteq S$$

  for every N in a fixed countable basis for $\mathbb{R}$. It follows that $f = \text{id}$ off of S, hence $X \cap Y \neq \emptyset$, a contradiction. □

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If no, then we've found a nontrivial automorphism of $\mathscr{P}(\omega_1)/\text{ctble}$. If yes, then $\mathscr{P}(\omega)/\text{fin} \simeq \mathscr{P}(\omega_1)/\text{fin}$ implies that every automorphism of $\mathscr{P}(\omega_1)/\text{ctble}$ is trivial.

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Theorem (Fleissner-Miller, 1980)

It is consistent with ZFC that there exists a Q-set X of size ω_1 such that $X \cup \mathbb{Q}$ is not a Q-set.

(It's easy to see that $X \cup \mathbb{Q}$ is a Q_B set, though.)

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Theorem (Larson-M.)

Suppose π is an automorphism of $\mathscr{P}(2^\kappa)/\mathscr{I}_{\kappa^+}$ which is trivial on every set of size κ^+. Then π is trivial.

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Remark

The function $f : 2^\kappa \to 2^\kappa$ defined in the proof is definable for any automorphism π of $\mathscr{P}(2^\kappa)/\mathscr{I}_{\kappa^+}$, and witnesses the triviality of π on a certain κ-complete ideal. The hypothesis on π is used to show that this ideal is everything.

Question

Is there, consistently, an isomorphism $\mathscr{P}(\omega_1)/\operatorname{fin} \to \mathscr{P}(\omega)/\operatorname{fin}$ which is trivial on every countable subset of ω_1?

Thank you!