Turning ternary relations into antisymmetric betweenness relations

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TOPOSYM 2016

The concept of betweenness

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- Natural to regard such a relation as a ternary predicate $[a, b, c]$, where $(a, b, c) \in X^3$.
- Birkhoff (1948) defined the betweenness relation $[\cdot, \cdot, \cdot]_o$ on a partially ordered set $(X, \leqslant)$ as an extension of that given above.

Examples of betweenness: partial orders

Definition

In a partially ordered set $(X, \leqslant)$ with $d \leqslant e \in X$, define the order interval $[d, e]_o = \{x \in X : d \leqslant x \leqslant e\}$.

- If each pair of elements in X has a common lower bound and a common upper bound in X, then say that $[a, b, c]_o$ if b belongs to *each* order interval that also contains a and c.

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- Metric space (X, d) (1928):
  define $[a, c, b]_M$ if $d(a, c) + d(c, b) = d(a, b)$.
- Natural alliance between intervals $[a, b]$ and ternary predicates $[a, c, b]$, in that we intend $c \in [a, b]$ iff $[a, c, b]$.

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while the largest is $X_\top := X^3 \setminus \{[a, b, a] \mid a \neq b\}$.

Bankston's insight: road systems

Definition

A road system on a nonempty set X is a family $\mathcal{R}$ of nonempty subsets (*roads*) of X such that

(i) $\{a\} \in \mathcal{R}$ for all $a \in X$,

(ii) for all $a, b \in X$, there is $R \in \mathcal{R}$ such that $a, b \in R$.

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Define $[a, c]_{\mathcal{R}} = \bigcap\{R \in \mathcal{R} : a, c \in R\}$.

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Theorem (Bankston, 2011)

A ternary relation $[\cdot,\cdot,\cdot]$ *on a set* X *can be generated from a road system if and only if* $[\cdot,\cdot,\cdot]$ *is an R-relation.*

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Antisymmetric R-relations

A road system $\mathcal{R}$ is *separative* if for any $a, b, c \in X$ with $b \neq c$, there is some $R \in \mathcal{R}$ such that either $a, b \in R$ and $c \notin R$ or $a, c \in R$ and $b \notin R$.

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monotone functions: for objects $(X, [\cdot,\cdot,\cdot]_X)$ and $(Y, [\cdot,\cdot,\cdot]_Y)$ then $f : X \to Y$ is a morphism provided $[a, b, c]_X \Rightarrow [f(a), f(b), f(c)]_Y$.

Some notation and definitions

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The inclusion functor $\mathbf{R_1} \hookrightarrow \mathbf{T}$

(R1) Reflexivity: $[a, b, b]$

The left adjoint is given by $(X, [\cdot,\cdot,\cdot]) \mapsto (X, [\cdot,\cdot,\cdot]')$ where

$$[\cdot,\cdot,\cdot]' = [\cdot,\cdot,\cdot] \cup \{[a, b, b] \in [\cdot,\cdot,\cdot] \mid a, b \in X\}.$$

Denote by L_1.

The inclusion functor $\mathbf{R_2} \hookrightarrow \mathbf{T}$

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The left adjoint exists - and is more involved. Call it L_3.

The inclusion functor $\mathbf{R_4} \hookrightarrow \mathbf{T}$

(R4) Transitivity: $[a, b, c] \wedge [a, d, c] \wedge [b, x, d] \Rightarrow [a, x, c]$

Has a left adjoint - call it L_4.

Adjoints as operators

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In fact, $L_4 \circ L_3 \circ L_1 \circ L_2$ defines the left adjoint to $\mathbf{R} \hookrightarrow \mathbf{T}$.

The subcategory **A** of antisymmetric R-relations

Antisymmetry: $[a, b, c] \wedge [a, c, b] \implies b = c$.

Question: does the inclusion functor $\mathbf{A} \hookrightarrow \mathbf{R}$ have a left adjoint?

Yes - demanding a change of underlying set; call it L_A.

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Theorem

The left adjoint is the direct limit of applying L_4 after L_A ω-many times.

Mar fhocal scoir

Given a lattice $(X, \leqslant)$, define $[a, b]_L = \{x : a \wedge b \leqslant x \leqslant a \vee b\}$.

Lemma

Let $(X, [\cdot, \cdot, \cdot])$ *be the R-relation generated from the lattice intervals (roads) described above.*

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Then $(X, \leqslant)$ *is distributive if and only if* $(X, [\cdot, \cdot, \cdot])$ *is antisymmetric.*