Connectedness and inverse limits with set-valued functions on intervals

Sina Greenwood

Coauthors: Judy Kennedy and Michael Lockyer

July 25, 2016

Outline

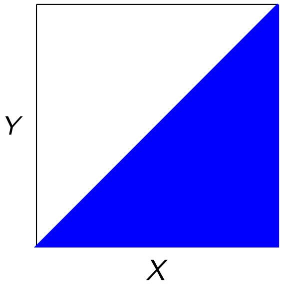
- CC-sequences and components bases
- Applications of component bases
- Large and small components
- The number of components

Definitions and notation

- $\mathbb{N} = \{0, 1, \dots\}$.
- 2^Y denotes the collection of non-empty closed subsets of Y.
- The graph of a function $f : X \to 2^Y$ is the set

$$\Gamma(f) = \{\langle x, y \rangle : y \in f(x)\}.$$

For example: $X = Y = [0, 1]$ and $f(x) = \{y : 0 \leq y \leq x\}$



- f is surjective if $f(X) = Y$.

- (Ingram, Mahavier) Suppose $f : X \to 2^Y$ is a function. If X and Y are compact Hausdorff spaces, then f is *upper semi-continuous (usc)* if and only if the graph of f is a closed subset of $X \times Y$.

For each $i \in \mathbb{N}$:
$\{X_i : i \in \mathbb{N}\}$ is a collection of compact Hausdorff spaces
$f_{i+1} : X_{i+1} \to 2^{X_i}$ is an usc function.

- The *generalised inverse limit (GIL)* of the sequence $\mathbf{f} = (X_i, f_i)_{i \in \mathbb{N}}$, denoted $\varprojlim \mathbf{f}$, is the set

$$\left\{ (x_n) \in \prod_{i \in \mathbb{N}} X_i : \forall\, n \in \mathbb{N}, x_i \in f_{i+1}(x_{i+1}) \right\}.$$

- The functions f_i are called *bonding maps*.
- We are interested in the case where each space $X_i = [0, 1]$, denoted I_i.

Definition

If $I = [0,1]$ and f is an upper semicontinuous surjective function from I into 2^I and has a connected graph, then we say that f is *full*.

If for each $i \in \mathbb{N}$, $I_i = [0,1]$, $\mathbf{f}$ is a sequence of functions $f_{i+1} : I_{i+1} \to 2^{I_i}$ and each f_{i+1} is full, then the sequence $\mathbf{f}$ is *full*.

Notation

1. If $m, n \in \mathbb{N}$ and $m \leq n$ then $[m, n] = \{i \in \mathbb{N} : m \leq i \leq n\}$.
2. π_j denotes the projection to I_j.
3. $\pi_{i,i-1}$ denotes the projection to $I_i \times I_{i-1}$ (usually to the graph if f_i).

Definition

Suppose that $\mathbf{f}$ is a full sequence, $m, n > 1$, and for each $i \in [m, n]$, $T_i \subseteq \Gamma(f_i)$. Then the *Mahavier product* of $T_m, \ldots, T_n$ is the set:

$$\left\{ \langle x_0, \ldots, x_n \rangle \in \prod_{i \le n} I_i : \forall\, i < n, \langle x_{i+1}, x_i \rangle \in T_{i+1} \right\},$$

denoted by $T_m \star \cdots \star T_n$ or by $\bigstar_{i \in [m,n]} T_i$.

Observe that

$$\bigstar_{i\in[m,n]}\Gamma(f_i)$$

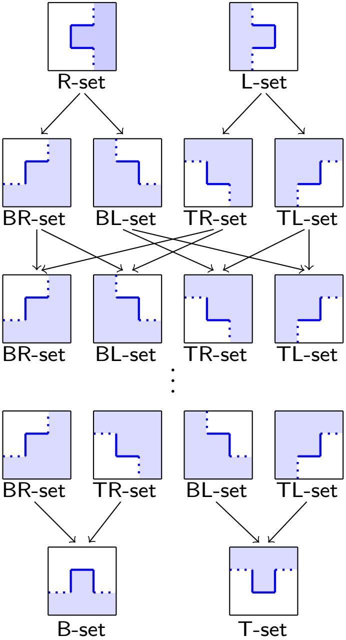
$$= \left\{ \langle x_0, \ldots, x_n \rangle \in \prod_{i\leq n} I_i : \forall\, i < n, \langle x_{i+1}, x_i \rangle \in \Gamma(f_{i+1}) \right\}$$

$$= \left\{ \langle x_0, \ldots, x_n \rangle \in \prod_{i\leq n} I_i : \forall\, i < n, x_i \in f_{i+1}(x_{i+1}) \right\}.$$

CC-sequences and component bases

Theorem (Greenwood and Kennedy)

Suppose $\mathbf{f}$ *is full. Then the system* $\mathbf{f}$ *admits a CC-sequence if and only if* $\varprojlim \mathbf{f}$ *is disconnected.*


R-set
L-set
BR-set
BL-set
TR-set
TL-set
BR-set
BL-set
TR-set
TL-set
BR-set
TR-set
BL-set
TL-set
B-set
T-set

Example

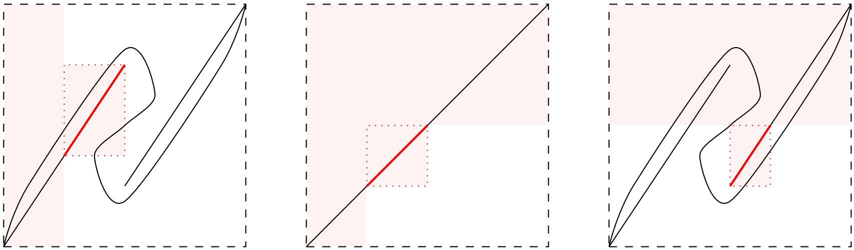
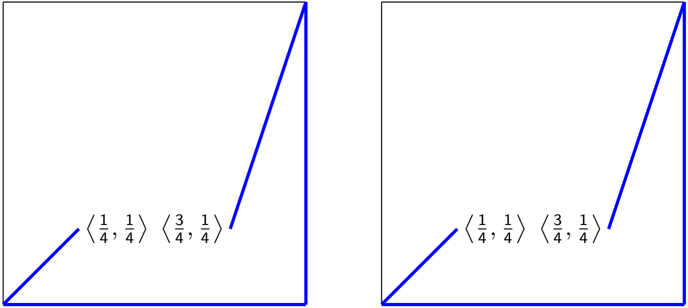


Figure: A weak component base: S_1 an L-set, S_2 a TL-set, S_3 a T-set.

Classic example



Any L-set must contain the point $\langle \frac{1}{4}, \frac{1}{4} \rangle$ and is not unique.
The singleton $\{\langle \frac{1}{4}, \frac{1}{4} \rangle\}$ is itself an L-set.
For any x, $0 < x < \frac{1}{4}$, the straight line from $\langle x, x \rangle$ to $\langle \frac{1}{4}, \frac{1}{4} \rangle$ is an L-set. Similarly for T-sets.
For example: $\left\{\left\langle \frac{1}{4}, \frac{1}{4} \right\rangle, \left\langle \frac{3}{4}, \frac{1}{4} \right\rangle\right\}$ is a component base.

Theorem

If $\mathbf{f}$ *is full then following statements are equivalent:*

1. *the system* $\mathbf{f}$ *admits a CC-sequence;*
2. *the system* $\mathbf{f}$ *admits a weak component base;*
3. *the system* $\mathbf{f}$ *admits a component base;*
4. $\varprojlim \mathbf{f}$ *is disconnected;*
5. *there exists* $n > 0$ *such that for every* $k \geq n$, $\bigstar_{i\in[1,k]}\Gamma(f_i)$ *is disconnected.*

Theorem

If $\mathbf{f}$ is a full sequence, C is a component of $\mathbf{f}$, $\langle S_m, \ldots, S_n \rangle$ is a weak component base, and

$$\pi_{[m-1,n]}(C) \cap \bigstar_{i \in [m,n]} S_i \neq \emptyset,$$

then

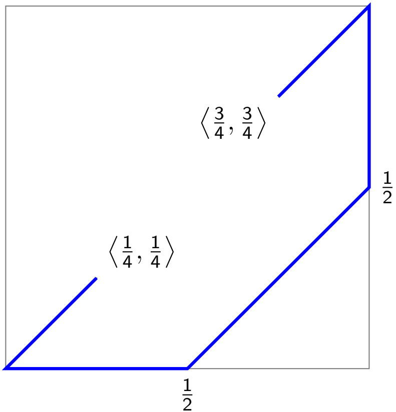
$$\pi_{[m-1,n]}(C) \subseteq \bigstar_{i \in [m,n]} S_i.$$

Definition

If $\mathbf{f}$ is a full sequence, $\sigma = \langle S_m, \ldots, S_n \rangle$ is a component base, and C is a component of $\varprojlim \mathbf{f}$ such that

$$\pi_{[m-1,n]}(C) = \bigstar_{i \in [m,n]} S_i,$$

then C is *captured* by $\langle S_m, \ldots, S_n \rangle$.



$S_1 = \{\langle \frac{1}{4}, \frac{1}{4} \rangle\}$ is an L-set.

$S_2 = \{\langle \frac{3}{4}, \frac{1}{4} \rangle\}$ is a TL-set.

$S_3 = \{\langle \frac{3}{4}, \frac{3}{4} \rangle\}$ is a T-set.

$\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \rangle \in S_1 \star S_2 \star S_3$.

$\langle S_1, S_2, S_3 \rangle$ is a component base.

Applications of CC-sequences

Theorem

If for each $\in \mathbb{N}$, $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a full bonding function and moreover each function f_{i+1} is continuous, then $\varprojlim \mathbf{f}$ is connected, and for each $n > 0$, $\bigstar_{i \in [1,n]} \Gamma(f_i)$ is connected.

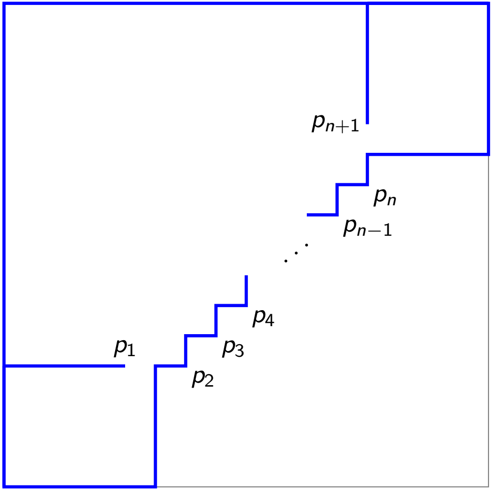
Proof.

No L-sets or R-sets.

□

For each n there is a single full bonding function f such that $\bigstar_{[1,n]}\Gamma(f)$ is connected and $\bigstar_{[1,n+1]}\Gamma(f)$ is disconnected.

Ingram gave examples of of such functions.
We give a new example using component bases.



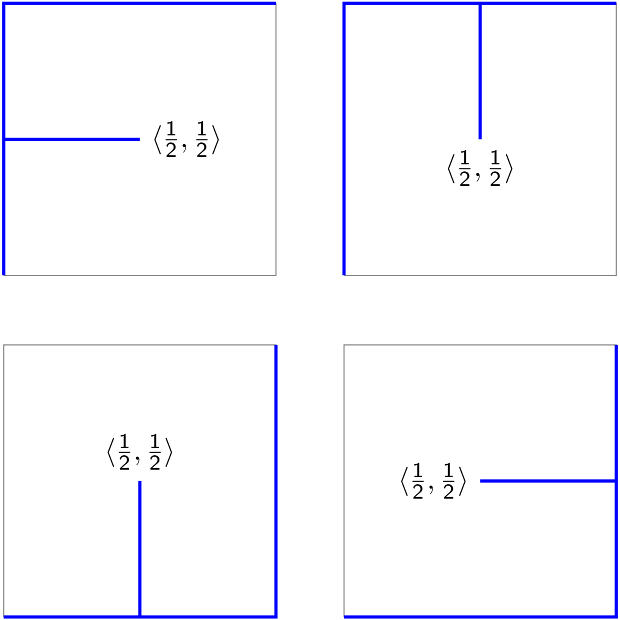
p_{n+1}
p_n
p_{n-1}
p_4
p_1
p_3
p_2

Problem (Ingram)

Suppose $\mathbf{f}$ *is a sequence of surjective upper semicontinuous functions on* $[0,1]$ *and* $\varprojlim \mathbf{f}$ *is connected. Let* $\mathbf{g}$ *be the sequence such that* $g_i = f_i^{-1}$ *for each* $i \in \mathbb{N}$*. Is* $\varprojlim \mathbf{g}$ *connected?*

Ingram and Marsh gave a full sequence $\mathbf{f}$ such that $\varprojlim \mathbf{f}$ is connected, and $\varprojlim (\mathbf{f}^{-1})$ is disconnected.

The problem is also discussed by Banič and Črepnjak.
Here is a new example:



There are no L-sets or R-sets in $\Gamma\left(f_1^{-1}\right)$.

What if there is a single bonding function?

Theorem

An inverse limit with a single full bonding function f is connected if and only if the inverse limit with single bonding function f^{-1} is connected.

Proof.

Suppose $\varprojlim \mathbf{f}$ is disconnected.

Then $\bigstar_{i\in[1,n]}\Gamma(f_i)$ is disconnected for some n.

So there exists a component base $\langle S_1, \ldots, S_n \rangle$.

Then $\langle S_n^{-1}, \ldots, S_1^{-1} \rangle$ is a component base of the system $\mathbf{f^{-1}}$.

The converse follows since $(f^{-1})^{-1} = f$. □

Large and small components

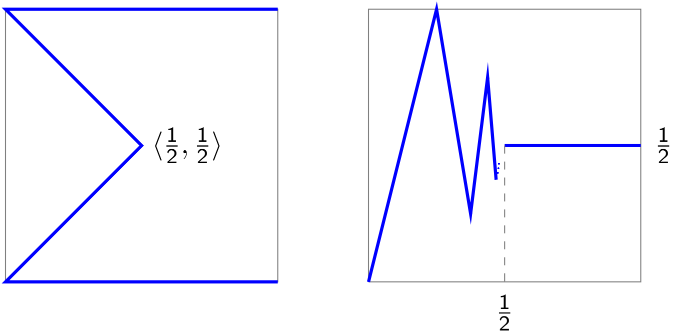
Banič and Kennedy showed that for every full sequence $\mathbf{f}$, $\varprojlim \mathbf{f}$ has at least one component C such that for every $i \in \mathbb{N}$, $\pi_{i+1,i}(C) = \Gamma(f_i)$.

Definition

Suppose $\mathbf{f}$ is a full sequence and C is a component of $\varprojlim \mathbf{f}$. Then C is *large* if for each $i \in \mathbb{N}$, $\pi_{i+1,i}(C) = \Gamma(f_{i+1})$, and C is *small* if it is not large.

If $m, n > 1$ and for each $i \in [m, n]$, $T_i \subseteq \Gamma(f_i)$, then D is a *large* component of $\bigstar_{i\in[m,n]}(T_i)$ if for each $i \in \mathbb{N}$, $\pi_{i+1,i}(D) = T_{i+1}$.

If $\mathbf{f}$ is a full sequence and C is a small component of $\varprojlim \mathbf{f}$, then it need not be the case that C is weakly captured by a component base.



$$C = \left\{ \left\langle \tfrac{1}{2}, \tfrac{1}{2}, x \right\rangle : x \in \left[\tfrac{1}{2}, 1 \right] \right\}$$

Theorem

For every full sequence $\mathbf{f}$, *if* $\varprojlim \mathbf{f}$ *has a small component* C *that is not captured by a component base, then the collection of captured components has a limit point in* C.

Theorem

For every full sequence $\mathbf{f}$, $\varprojlim \mathbf{f}$ *has exactly one large component.*

Corollary

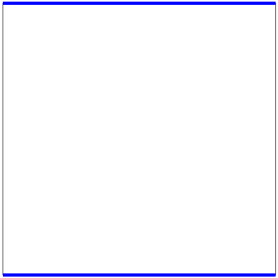
If $\varprojlim \mathbf{f}$ *is disconnected then it has a small component.*

The number of components of an inverse limit

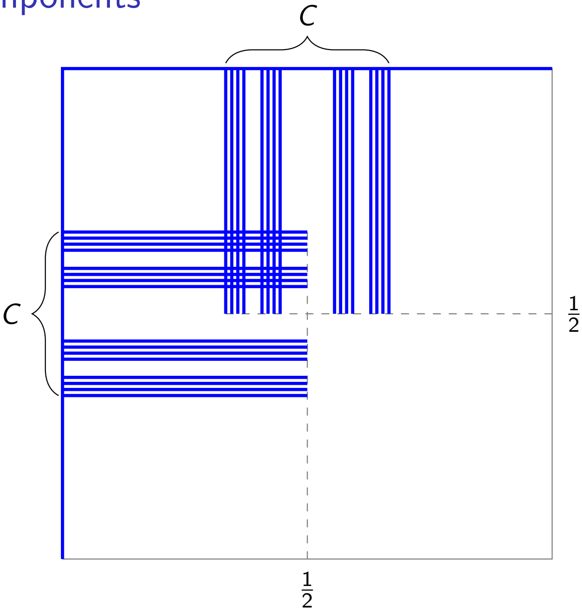
Theorem

An inverse limit with a single upper semicontinuous function whose graph is the union of two maps without a coincidence point has c many components.

Perhaps the most extreme example is:



$\mathfrak{c}$ many components



For every $c \in C$, $\left\{\left\langle \frac{1}{2}, c \right\rangle, \left\langle c, \frac{1}{2} \right\rangle\right\}$ is a component base.

In the previous example, the inverse limit has $\mathfrak{c}$ many components, and so do each of the Mahavier products of $\mathbf{g}$.

In this example $\varprojlim \mathbf{f}$ has $\mathfrak{c}$ many components, but every Mahavier product has only finitely many components.

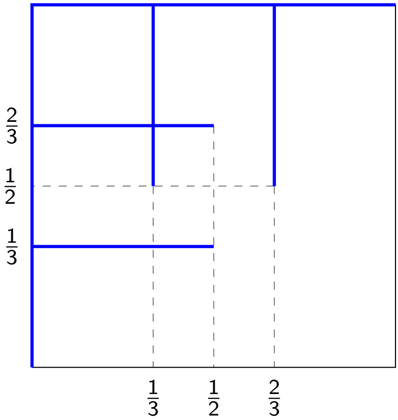


Figure:

In the previous example the sequence admitted infinitely many component bases

It is possible that a full sequence $\mathbf{f}$ has a finite number of components bases, but $\varprojlim \mathbf{f}$ has $\mathfrak{c}$ many components.

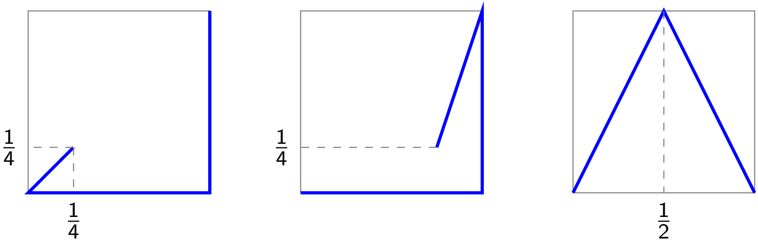


Figure:

Thank you