Topological entropy on totally disconnected locally compact groups

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(joint work with Simone Virili)

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Topological entropy (h_{top})

- [Adler, Konheim, McAndrew 1965]:
  for continuous selfmaps of compact spaces.
- [Bowen 1971]:
  for uniformly continuous selfmaps of metric spaces.
- [Hood 1974]:
  for uniformly continuous selfmaps of uniform spaces.
- We consider it:
  for continuous endomorphisms of locally compact groups.
- These entropies coincide on compact groups.
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Let G be a locally compact group, μ a Haar measure on G, $\mathcal{C}(G)$ the family of all compact neighborhoods of 1 in G, $\phi : G \to G$ a continuous endomorphism.

- For $n > 0$, the *n-th ϕ-cotrajectory of $U \in \mathcal{C}(G)$* is

$$C_n(\phi, U) = U \cap \phi^{-1}(U) \cap \ldots \cap \phi^{-n+1}(U) \in \mathcal{C}(G).$$

- The *topological entropy* of ϕ with respect to $U \in \mathcal{C}(G)$ is

$$\boxed{H_{top}(\phi, U) = \limsup_{n\to\infty} \frac{-\log \mu(C_n(\phi, U))}{n}.}$$

(It does not depend on the choice of the Haar measure μ.)

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$$\boxed{h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{C}(G)\}.}$$

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Problem (Additivity of topological entropy)

Let G be a locally compact group, $\phi : G \to G$ a continuous endomorphism and N a ϕ-invariant closed normal subgroup of G. Is it true that

$$h_{top}(\phi) = h_{top}(\phi \restriction_N) + h_{top}(\bar{\phi}),$$

where $\bar{\phi} : G/N \to G/N$ is the endomorphism induced by ϕ?

$$\begin{array}{ccccc} N & \longrightarrow & G & \longrightarrow & G/N \\ \downarrow{\scriptstyle \phi\restriction_N} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \bar{\phi}} \\ N & \longrightarrow & G & \longrightarrow & G/N \end{array}$$

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We consider the case when
G is a totally disconnected locally compact group
and $\phi : G \to G$ is a continuous endomorphism.

Let $\mathcal{B}(G) = \{U \leq G : U \text{ compact, open}\}$.

[van Dantzig 1931]: $\mathcal{B}(G)$ is a base of the neighborhoods of 1 in G.

[Dikranjan-Sanchis-Virili 2012]:

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{B}(G)\};$$

moreover, for $U \in \mathcal{B}(G)$,

$$H_{top}(\phi, U) = \lim_{n\to\infty} \frac{\log[U : C_n(\phi, U)]}{n}.$$

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Let G be a totally disconnected locally compact group and $\phi : G \to G$ a continuous endomorphism.

For $U \in \mathcal{B}(G)$, let:

- $U_0 = U$;
- $U_{n+1} = U \cap \phi(U_n)$ for every $n > 0$;
- $U_+ = \bigcap_{n=0}^{\infty} U_n$.

Then:

- $U_{n+1} \subseteq U_n$ for every $n > 0$;
- U_+ is a compact subgroup of G such that $U_+ \subseteq \phi(U_+)$.

Theorem (Limit-free formula; GB-Virili 2016)

$$H_{top}(\phi, U) = \log[\phi(U_+) : U_+],$$

[GB 2015]: for topological automorphisms.

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Theorem (Addition Theorem; GB-Virili 2016)

If $\ker \phi \subseteq N$ *and* $\phi(N) = N$, *then*

$$h_{top}(\phi) = h_{top}(\phi \restriction_N) + h_{top}(\bar{\phi}).$$

$\ker \phi \subseteq N$ and $\phi(N) = N$ if and only if
$\phi \restriction_N$ is injective and $\bar{\phi}$ is surjective.

Corollary

If $\phi : G \to G$ *is a topological automorphism, then*

$$h_{top}(\phi) = h_{top}(\phi \restriction_N) + h_{top}(\bar{\phi}).$$

Let G be a totally disconnected locally compact group, $\phi : G \to G$ a continuous endomorphism.

If $N \leq G$ compact (not necessarily normal) with $\phi(N) = N$, then $G/N = \{xN : x \in G\}$ is a locally compact uniform space and $\bar{\phi} : G/N \to G/N$ is a uniformly continuous map.

Then

$$h_{top}(\bar{\phi}) = \sup\{H_{top}(\phi, U) : N \subseteq U \in \mathcal{B}(G)\}.$$

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[Willis 2015] (in 2001 for topological automorphisms):

- The *scale* of a continuous endomorphism $\phi : G \to G$ of a totally disconnected locally compact group G is

$$s(\phi) = \min\{[\phi(U) : U \cap \phi(U)] : U \in \mathcal{B}(G)\}.$$

- $U \in \mathcal{B}(G)$ is *minimizing* if $s(\phi) = [\phi(U) : U \cap \phi(U)]$.
- $\mathrm{nub}\,\phi \leq G$ is compact and $\phi(\mathrm{nub}\,\phi) = \mathrm{nub}\,\phi$;

$$\mathrm{nub}\,\phi := \bigcap\{U \in \mathcal{B}(G) : U \text{ minimizing}\}.$$

Theorem (GB-Virili 2016)

For $\bar{\phi} : G/\mathrm{nub}\,\phi \to G/\mathrm{nub}\,\phi$ *the map induced by* ϕ,

$$h_{top}(\bar{\phi}) = \log s(\phi).$$

Corollary (Berlai-Dikranjan-GB and Spiga 2013)

$h_{top}(\phi) = \log s(\phi)$ *if and only if* $\mathrm{nub}\,\phi = \{1\}$.

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Algebraic entropy (h_{alg})

- [Adler, Konheim, McAndrew 1965; Weiss 1974; Dikranjan-Goldsmith-Salce-Zanardo 2009]:
  for endomorphisms of discrete (torsion) abelian groups.
- [Peters 1979]:
  for automorphisms of discrete abelian groups.
- [Dikranjan-GB 2012, 2016]:
  for endomorphisms of discrete (abelian) groups.
- [Peters 1981]:
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- For $n > 0$, the *n-th ϕ-trajectory of $U \in \mathcal{C}(G)$* is

$$T_n(\phi, U) = U \cdot \phi(U) \cdot \ldots \cdot \phi^{n-1}(U) \in \mathcal{C}(G).$$

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$$\boxed{H_{alg}(\phi, U) = \limsup_{n\to\infty} \frac{\log \mu(T_n(\phi, U))}{n}.}$$

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Let $\widehat{G}$ be the Pontryagin dual of G and $\widehat{\phi} : \widehat{G} \to \widehat{G}$ the dual of ϕ.
Then $\mathcal{B}(\widehat{G})$ is cofinal in $\mathcal{C}(\widehat{G})$. Hence,

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Theorem (Bridge Theorem; Dikranjan-GB 2014)

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}).$$

[Weiss 1974]: for totally disconnected compact abelian groups.
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