# Congruence-free compact semigroups

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# Plan

## Definition

An equivalence relation $\rho$ on a semigroup $S$ is called a *left congruence* if $a \, \rho \, b$ implies $ca \, \rho \, cb$ for all $a, b, c \in S$.
The notion of a *right congruence* is defined dually.

## Definition

An equivalence relation $\rho$ on a semigroup $S$ is said to be a *congruence* if $a \, \rho \, b, c \, \rho \, d$ implies $ac \, \rho \, bd$ for all $a, b, c, d \in S$.

## Fact

*An equivalence relation on a semigroup $S$ is a congruence if and only if it is a left congruence and a right congruence on $S$.*

*A congruence on a semigroup is not determined (in general) by any of its equivalence classes.*

- A classical result of semigroup theory says that a **finite** *congruence-free* semigroup $S$ (i.e., $S$ has exactly two congruences) without zero such that card$(S) > 2$ is a simple group; Tamura (1956).
- One of the problems that has given **impetus** to the theory of **topological semigroups** is the problem of finding topological and /or algebraic hypothesis on a semigroup which imply that it must be a group (Wallace (1955)).
- I have generalized the results of Tamura from the '**finite case**' to the '**compact case**'.

- Let $\rho$ be a congruence on a semigroup $S$. Then the quotient space $S/\rho = \{a\rho : a \in S\}$ is a semigroup with respect to the multiplication $(a\rho)(b\rho) = (ab)\rho$. Denote the natural morphism from $S$ onto $S/\rho$ by $\rho^\natural$, that is, $a\rho^\natural = a\rho$ $(a \in S)$.
- Let $S$ be a topological semigroup. A congruence on $S$ is called *topological* if $S/\rho$ is a topological semigroup with respect to the quotient topology

$$\mathcal{O}_{S/\rho} = \{U \subseteq S/\rho : U{\rho^\natural}^{-1} \in \mathcal{O}_S\}.$$

**Fact**

*A congruence on a compact semigroup $S$ is topological if and only if it is closed in the product topology $S \times S$.*

## Definition

A compact semigroup is said to be *congruence-free* if the set of its topological congruences is equal to $\{1_S, S \times S\}$.

## Theorem

*Every infinite congruence-free compact semigroup S is a connected metric Lie group (so all left and right translations of S are isometries) with cardinality* $\mathfrak{c}$.

I will present a sketch of the proof of the above theorem. For this we shall need some definitions and results.

- A semigroup is called a *left zero* semigroup if it satisfies the identity $xy = x$.
- A semigroup is called a *right zero* semigroup if it satisfies the identity $xy = y$.
- A direct product of any left zero semigroup and any right zero semigroup is called a *rectangular band*.
- Denote the set of *idempotents* of a semigroup $S$ by

$$E_S = \{e \in S : ee = e\}$$

and note that the relation $\leq$ defined on $E_S$ by

$$e \leq f \Leftrightarrow e = ef = fe$$

is a partial order on $E_S$ (the so-called *natural partial order* on $E_S$).

- A nonempty subset $A$ of a semigroup $S$ is said to be a *left ideal* of $S$ if $SA \subseteq A$.
  Note that $S^1a = Sa \cup \{a\}$ is the least *left ideal* of $S$ containing the element $a \in S$.
- A nonempty subset $A$ of a semigroup $S$ is said to be a *right ideal* of $S$ if $AS \subseteq A$.
  Note that $aS^1 = aS \cup \{a\}$ is the least *right ideal* of $S$ containing the element $a \in S$.
- A nonempty subset $A$ of a semigroup $S$ is said to be an *ideal* of $S$ if $SA \cup AS \subseteq A$.
  Note that $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$ is the least *ideal* of $S$ containing the element $a \in S$.

- Let $A$ be an ideal of a semigroup $S$. Then the relation

$$\rho_A = (A \times A) \cup 1_S,$$

where $1_S$ is the identity relation on $S$, is an algebraic congruence on $S$ (the so-called *Rees congruence*).

- Let $S$ be a semigroup, $a, b \in S$. Recall that

$$a \,\mathcal{L}\, b \Leftrightarrow S^1 a = S^1 b, \quad a \,\mathcal{R}\, b \Leftrightarrow aS^1 = bS^1,$$

$$a \,\mathcal{J}\, b \Leftrightarrow S^1 a S^1 = S^1 b S^1$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \quad \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \vee \mathcal{R}.$$

These equivalence relations, known under the name of Green's relations, have played a fundamental role in the development of semigroup theory. Note that $\mathcal{D} \subseteq \mathcal{J}$ and denote for any $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$ the equivalence $\mathcal{K}$-class containing $a$ by $\mathcal{K}_a$.

- Recall that Green's Theorem says that in an arbitrary semigroup $S$, either $\mathcal{H}_a \cap \mathcal{H}_a^2 = \emptyset$ or $\mathcal{H}_a$ is a group. In particular, $\mathcal{H}_e$ is a group for any $e \in E_S$.
- Each $\mathcal{D}$-class in a semigroup $S$ is a union of $\mathcal{L}$-classes, and also a union of $\mathcal{R}$-classes. The intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class is either empty or is an $\mathcal{H}$-class. As $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$,

$$a \mathcal{D} b \iff \mathcal{R}_a \cap \mathcal{L}_b \neq \emptyset \iff \mathcal{L}_a \cap \mathcal{B}_b \neq \emptyset.$$

  Hence it is convenient to visualize a $\mathcal{D}$-class as what Clifford and Preston (1961) have called an 'eggbox', in which each row represents an $\mathcal{R}$-class, and each column represents an $\mathcal{L}$-class, and each cell represents an $\mathcal{H}$-class.

- Let $e, f$ be idempotents of a semigroup $S$ such that $e \mathcal{R} f$, that is, $eS = fS$. Then $e \in eS = fS$. Hence $fe = e$, so:

**Fact**

*In an arbitrary $\mathcal{R}$-class $R$ of a semigroup $S$, $E_S \cap R$ is either empty or is a right zero semigroup.*

**Definition**

A semigroup $S$ with $E_S \neq \emptyset$ is called *completely simple* if $\mathcal{D} = S \times S$ and every idempotent of $S$ is minimal with respect to the natural partial order $\leq$, that is, $\leq = 1_S$.

The following important theorem will be useful.

**Theorem**

*A semigroup $S$ is completely simple if and only if $\mathcal{H}$ is a congruence on $S$ such that $S/\mathcal{H}$ is a rectangular band.*

**Theorem**

*Each of Green's relation is closed in an arbitrary compact semigroup.*

**Theorem**

*Each compact semigroup has a least ideal which is a completely simple compact semigroup.*

## The proof of the main theorem.

- Recall that if $S$ is a compact semigroup and $A \notin \{\emptyset, S\}$ is an open subset of $S$ which is simultaneously closed in $S$, then the relation

$$\rho = \{(a, b) \in S \times S : (\forall x, y \in S^1)(xay \in A \Leftrightarrow xby \in A)\}$$

  is an algebraic congruence on $S$ such that every $\rho$-class of $S$ is **open** in $S$.
- Notice that $\rho \subseteq \tau^A$, where $\tau^A$ is the equivalence on $S$ induced by the partition $\{A, S \setminus A\}$ of $S$.
- As $S$ is compact, $S/\rho$ must be finite, say

$$S/\rho = \{a_1\rho, a_2\rho, \ldots, a_n\rho\},$$

  and so every $\rho$-class of $S$ is also **closed** in $S$.

- Thus the relation

$$\rho = (a_1\rho \times a_1\rho) \cup (a_2\rho \times a_2\rho) \cup \cdots \cup (a_n\rho \times a_n\rho)$$

is closed in $S \times S$. Consequently, $\rho$ is a topological congruence on $S$ and $\rho \neq S \times S$.

- If in addition, the compact semigroup $S$ is congruence-free, then $\rho = 1_S$, so $S$ is finite, therefore, if $S$ is an infinite congruence-free compact semigroup, then $S$ must be **connected**, and since $S$ is a Tychonoff space, $S$ has cardinality not less than $\mathfrak{c}$.
- I have also proved that if a compact semigroup with 0 is congruence-free, then the set $\{0\}$ is open. Thus every **infinite** congruence-free compact semigroup has **no** 0.

- Let $S$ be an infinite congruence-free compact semigroup. Then $S$ has a closed ideal $A$ which is completely simple. Hence the Rees congruence $\rho_A$ is topological. As $S$ is congruence-free, $\rho_A \in \{1_S, S \times S\}$. Note that $\rho_A = 1_S$ implies that $S$ has 0 (the only element of $A$). Thus $\rho_A = S \times S$. Consequently, $S = A$ is a completely simple semigroup.
- Hence $\mathcal{H}$ is a closed congruence on $S$ such that $S/\mathcal{H}$ is a rectangular band. Thus $\mathcal{H} = 1_S$ or $\mathcal{H} = S \times S$.

- If $\mathcal{H} = 1_S$, then $S$ is a rectangular band, and then the both relations $\mathcal{L}$ and $\mathcal{R}$ are closed congruences on $S$ and so they are both topological congruences on $S$. If $\mathcal{L} = \mathcal{R} = 1_S$, then $S$ is the trivial semigroup, a contradiction with the assumption of the theorem. Similarly, $\mathcal{L} = \mathcal{R} = S \times S$ implies that $S$ is trivial. Consequently, $S$ is either a left zero semigroup or a right zero semigroup. Let $a, b \in S$ be such that $a \neq b$. As $S$ is infinite, the relation

$$\rho = (\{a, b\} \times \{a, b\}) \cup 1_S$$

is a proper algebraic congruence on $S$. Clearly, $\rho$ is closed in $S \times S$. Hence $\rho$ is topological but this is not possible.

- Thus $\mathcal{H} = S \times S$, so $S$ is a group (by Green's Theorem).
- The Ellis' Theorem (*a semitopological locally compact semigroup which is a group must be a topological group, that is, the operation of taking inverses is continuous*) implies that $S$ is a compact group.
- According to Morikuni, in 1953 Yamabe obtained the final answer to Hilbert's Fifth Problem. Namely, he showed that a connected locally compact group *without small subgroups* is a Lie group.
- Recall that a compact group has no small subgroups if it has no small normal subgroups.

- Note that if $A$ is a normal subgroup of a compact group $G$, then cl$A$ is a closed normal subgroup of $G$, therefore, if $G$ is congruence-free and $A \neq \{1\}$, then cl$A = G$.
- The above implies that every infinite congruence-free compact semigroup is a Lie group.

- Recall that a metric $m$ on a semigroup $S$ is *subinvariant* if for all $a, b, c \in S$ we have

$$m(ca, cb) \leq m(a, b), \quad m(ac, bc) \leq m(a, b).$$

Notice that if $S$ is a group, then

$$m(ca, cb) = m(a, b), \quad m(ac, bc) = m(a, b)$$

for each $a, b, c \in S$, that is, all left and right translations of $S$ are isometries.
Also, a topological semigroup $S$ is called a *metric semigroup* if there exists a subinvariant metric on $S$ which determines the topology of $S$.

- Recall that if $\varphi_{a,b}$ is a continuous function from a compact semigroup $S$ into $[0,1]$ such that $a\varphi_{a,b} = 0$ and $b\varphi_{a,b} = 1$, then the relation

$$\rho_{\varphi_{a,b}} = \{(a,b) \in S \times S : (\forall x, y \in S^1)\,(xay)\varphi_{a,b} = (xby)\varphi_{a,b}\}$$

is a closed congruence on $S$ such that $S/\rho_{\varphi_{a,b}}$ is a metric semigroup.
Recall that the subinvariant metric $m$ on $S/\rho_{\varphi_{a,b}}$ is defined by

$$m(a\rho_{\varphi_{a,b}}, b\rho_{\varphi_{a,b}}) = \sup\{|(xay)\varphi_{a,b} - (xby)\varphi_{a,b}| : x, y \in S^1\}.$$

- Clearly, $(a, b) \notin \rho_{\varphi_{a,b}}$. Consequently, if $S$ is congruence-free, then $\rho_{\varphi_{a,b}} = 1_S$ and so $S$ is a metric semigroup.
- Finally, $S$ has cardinality $\mathfrak{c}$ by a little part of the celebrated result of Professor Arkhangel'skii (1969). Namely:

**Theorem**

*For every infinite compact space $X$ we have card$(X) \leq 2^{\chi(X)}$.*

In group theory, a *simple* Lie group is a connected locally compact non-Abelian Lie group G which does not have nontrivial *connected* normal subgroups. Clearly, the well-known classification of simple Lie groups has nothing to do with the classification of finite simple groups.
On the other hand, it is easy to see that a compact group is congruence-free if and only if it does not have nontrivial *closed* normal subgroups.

**Problem**

*Classify all congruence-free compact groups.*