Topological and Algebraic Entropy for Locally Linearly Compact Vector Spaces

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joint work with A. Giordano Bruno

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Overview

Linearly Compact Spaces

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Definition (Lefschetz, 1942)

A linearly topologized space V is linearly compact if, and only if, any collection of closed linear varieties of V with the finite intersection property has non-empty intersection.

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Proposition (Lefschetz, 1942)

Every linearly compact space V over a discrete field $\mathbb{K}$ is a product of one-dimensional spaces. In particular, V is compact if, and only if, $\mathbb{K}$ is finite.

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General properties

Let V be a linearly topologized space. Thus

(a) if W is a linearly compact subspace of V, then W is closed;

(b) if V is a linearly compact space, W a closed subspace of V, then W is linearly compact;

(c) linear compactness is preserved under continuous homomorphisms;

(d) a n.a.s.c. for a discrete V to be linearly compact is to have finite dimension. Hence every finite-dimensional V is linearly compact;

(e) the product of linearly compact spaces is linearly compact;

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An l.l.c. space is topologically isomorphic to $\bigoplus_{i\in I} \mathbb{K} \times \prod_{j\in J} \mathbb{K}$.

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Locally linearly compact spaces over FINITE fields

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For $n \in \mathbb{N}$ and $U \in \mathcal{B}(V) = \{$linearly compact open subspaces of $V\}$, let

$$C_n(\phi, U) = U \cap \phi^{-1} U \cap \phi^{-2} U \cap \ldots \cap \phi^{-n+1} U,$$

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$$H^*(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \dim\left(\frac{U}{C_n(\phi, U)}\right),$$

By mirroring the definition of the topological entropy for totally disconnected LC groups, which is based on the finite index of a partial cotrajectory into its compact open subgroup.

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Inspired by the definition of the algebraic entropy for compactly covered LC groups,
by the algebraic entropy (w.r.t. the dimension) for discrete vector spaces,
by the intrinsic entropy defined for (discrete) abelian groups

Comparison of entropies

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Remark

- Let V be an l.l.c. space over a finite field $\mathbb{F}$ and $\phi\colon V \to V$ a continuous endomorphism. Then

$$\mathrm{ent}(\phi) = \frac{1}{\log|\mathbb{K}|} \cdot \mathrm{h}_{alg}(\phi) \quad \text{and} \quad \mathrm{ent}^*(\phi) = \frac{1}{\log|\mathbb{K}|} \cdot \mathrm{h}_{top}(\phi),$$

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  where $\mathrm{h}_{alg}(\_)$ and $\mathrm{h}_{top}(\_)$ denote respectively the algebraic and topological entropy defined for totally disconnected LCA groups.

- Let V be a discrete vector space over an arbitrary field $\mathbb{K}$, and $\phi\colon V \to V$ a linear map. Then

$$\mathrm{ent}(\phi) = \mathrm{ent}_{\mathrm{dim}}(\phi).$$

A. Giordano Bruno, L. Salce, *A soft introduction to algebraic entropy*. Arab. J. Math. **1.1** (2012): 69-87.

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Proposition

Let $\phi\colon V \to V$ be a continuous endomorphism of an l.l.c. space V, then

1. *(Invariance under conjugation)*
   for every topological isomorphism $\alpha\colon V \to W$ of l.l.c. spaces, one has

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3. *(Monotonicity under restrictions and quotients)*
   for every closed linear subspace W *of* V *such that* $\phi W \leq W$ *one has*
   $$\mathrm{ent}(\phi) \geq \max\{\mathrm{ent}(\phi\restriction_W), \mathrm{ent}(\overline{\phi})\},$$
   $$\mathrm{ent}^*(\phi) \geq \max\{\mathrm{ent}^*(\phi\restriction_W), \mathrm{ent}^*(\overline{\phi})\},$$
   where $\overline{\phi}\colon V/W \to V/W$ *is induced by* ϕ*.*

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Fact (C., Giordano Bruno)

Let $\phi\colon V \to V$ be a continuous endomorphism of an l.l.c. space V, then

(a) $\mathrm{ent}(\phi) = 0$ *whenever V is linearly compact,*

(b) $\mathrm{ent}^*(\phi) = 0$ *whenever V is discrete.*

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$$\iota_* \colon V_* \to V, \quad p_* \colon V \to V_*, \quad * \in \{c, d\},$$

be the canonical injections and projections. Accordingly, we may associate to ϕ the following decomposition

$$\phi = \begin{pmatrix} \phi_{cc} & \phi_{dc} \\ \phi_{cd} & \phi_{dd} \end{pmatrix}.$$

Let $V = V_c \times V_d$ be an l.l.c. space and ϕ a continuous endomorphism. Let

$$\iota_* \colon V_* \to V, \quad p_* \colon V \to V_*, \quad * \in \{c, d\},$$

be the canonical injections and projections. Accordingly, we may associate to ϕ the following decomposition

$$\phi = \begin{pmatrix} \phi_{cc} & \phi_{dc} \\ \phi_{cd} & \phi_{dd} \end{pmatrix}.$$

Theorem (C., Giordano Bruono)

Let $\phi \colon V \to V$ be a continuous endomorphism of an l.l.c. space V. Then

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi_{dd}) \quad \textit{and} \quad \mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi_{cc}).$$

Limit free formula

Limit free formula (C., Giodano Bruno)

Let V be an l.l.c. space, $\phi : V \to V$ a continuous endomorphism. For every linearly compact open subspace U

- one has an open linear subspace U^- such that
$$H(\phi, U) = \dim\left(\frac{U^-}{\phi^{-1}U^-}\right),$$
- one has a linearly compact subspace U^+ of V such that
$$H^*(\phi, U) = \dim\left(\frac{\phi U^+}{U^+}\right).$$

Inspired by Willis' work concerning the scale function for totally disconnected LC groups.

Addition Theorem

Addition Theorem

Remark

We say that the *Addition Theorem (breifly, AT) holds for the topological entropy* ent^* *over l.l.c. spaces* if

$$\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi\restriction_W) + \mathrm{ent}^*(\overline{\phi}),$$

for every closed linear subspace W of V such that $\phi W \leq W$ and the map $\overline{\phi}\colon V/W \to V/W$ is induced by ϕ. Analogously, for ent.

Addition theorem: algebraic entropy

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Step 1 Reduction to discrete vector spaces.

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Theorem (C., Giordano Bruno)

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A. Giordano Bruno, L. Salce, A soft introduction to algebraic entropy. Arab. J. Math. ***1.1*** *(2012): 69-87.*

Addition theorem: topological entropy

PART I	PART II

Addition theorem: topological entropy

PART I

↓

Reduction to cont. endomorphisms of Linearly Compact spaces

PART II

Addition theorem: topological entropy

PART I

↓

Reduction to cont. endomorphisms of Linearly Compact spaces

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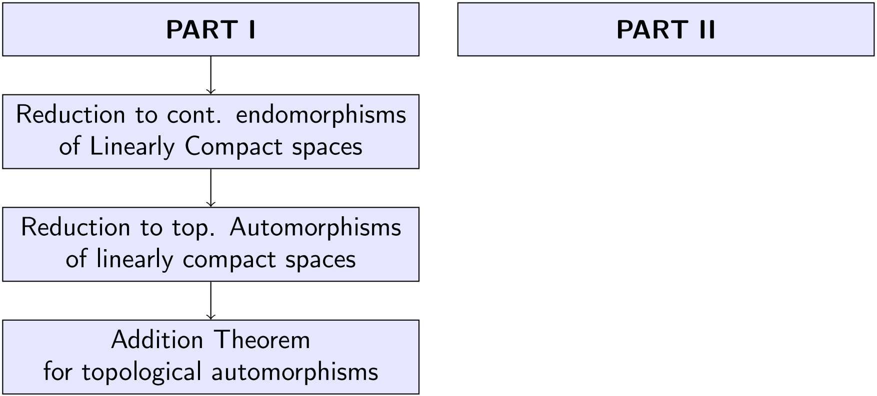
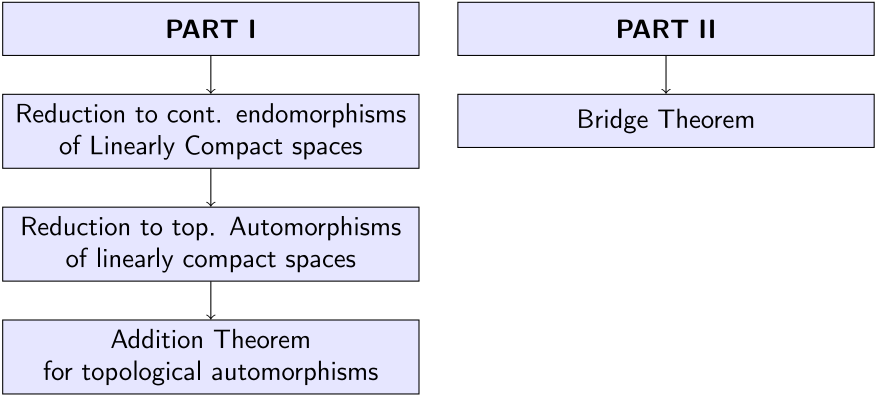
Reduction to top. Automorphisms of linearly compact spaces

PART II

Addition theorem: topological entropy



Addition theorem: topological entropy



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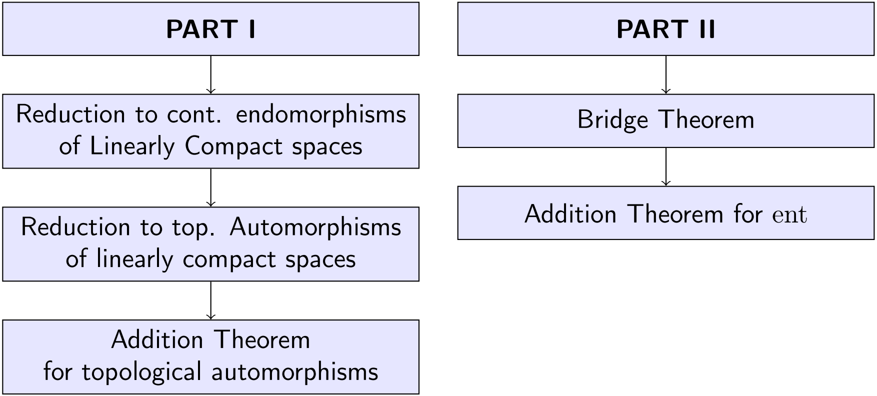
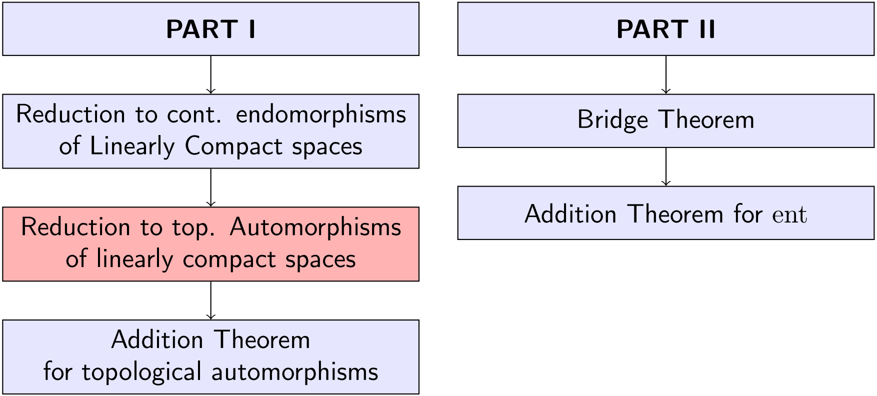


Addition theorem: topological entropy



Addition Theorem for linearly compact spaces - Part I

Reduction to automorphisms

1. For a continuous endomorphism $\phi\colon V \to V$ of a linearly compact space V, let $\mathcal{L}V$ denote the inverse limit of the following inverse system

$$\cdots \xrightarrow{\phi} V \xrightarrow{\phi} V \xrightarrow{\phi} \cdots \xrightarrow{\phi} V \xrightarrow{\phi} V_0 = V.$$

Addition Theorem for linearly compact spaces - Part I

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2. The natural map $\prod \phi\colon \prod_{n\in\mathbb{N}} V \to \prod_{n\in\mathbb{N}} V$ such that $\{x_n\} \mapsto \{\phi(x_n)\}$ induces a continuous endomorphism $\mathcal{L}\phi : \mathcal{L}V \to \mathcal{L}V$ making the following diagram

$$\begin{array}{ccc} \prod_{n\in\mathbb{N}} V & \xrightarrow{\prod \phi} & \prod_{n\in\mathbb{N}} V \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} \\ \mathcal{L}V & \xrightarrow[\mathcal{L}\phi]{} & \mathcal{L}V \end{array}$$

commute, where ι is the canonical embedding.

Addition Theorem - Part I

Proposition (C., Giordano Bruno)

For every continuous endomorphism ϕ of a linearly compact space V, one has that the following hold:

(a) $\mathcal{L}V$ *is a linearly compact space;*

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(c) $\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\mathcal{L}\phi)$.

L. Salce and S. Virili. *The addition theorem for algebraic entropies induced by non-discrete length functions.* Forum Mathematicum. 2015.

Addition theorem: topological entropy

PART I

- Reduction to Linearly Compact spaces
- Reduction to topological automorphisms
- Addition Theorem for topological automorphisms

PART II

- Bridge Theorem
- Addition Theorem for ent

Lefschetz Duality

Let $\mathrm{CHom}(\mathrm{V}, \mathbb{K})$ be the space of all continuous endomorphisms from V to $\mathbb{K}$.

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For a linear subspace A of V, the *annihilator of A in* $\mathrm{CHom}(\mathrm{V}, \mathbb{K})$ is defined by

$$A^{\perp} = \{\chi \in \mathrm{CHom}(\mathrm{V}, \mathbb{K}) : \chi(A) = 0\}.$$

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We denote by $\widehat{V}$ the vector space $\mathrm{CHom}(\mathrm{V}, \mathrm{K})$ endowed with the topology locally generated by

$$\{A^{\perp} \mid A \leq V,\ A \text{ linearly compact}\},$$

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which is an l.l.c. space.

In particular, V is discrete if, and only if, $\widehat{V}$ is linearly compact; V is linearly compact if, and only if, $\widehat{V}$ is discrete.

Remark

If $\dim(V) < \infty$, then $\widehat{V}$ coincide with classical dual space of V.

Lefschetz Duality

Theorem (Lefschetz, 1942)

An l.l.c. space V is topologically isomorphic to $\widehat{\widehat{V}}$.

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The dual functor $V \mapsto \widehat{V}$ defines a duality between the full subcategory of all linearly compact spaces and the full subcategory of all discrete vector spaces.

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The dual functor $V \mapsto \widehat{V}$ defines a duality between the full subcategory of all linearly compact spaces and the full subcategory of all discrete vector spaces.

Remark

This result is the analogue of Pontryagin-van Kampen duality in the context of locally linearly compact spaces.

Bridge Theorem

Theorem (C., Giordano Bruno)

Let V be an l.l.c. space and $\phi : V \to V$ a continuous endomorphism. Then, for every $U \in \mathcal{B}(V)$,

$$H^\star(\phi, U) = H(\widehat{\phi}, U^\perp),$$

where $\widehat{\phi}\colon \widehat{V} \to \widehat{V}$ is given by $\chi \to \chi \circ \phi$. Consequently,

$$\mathrm{ent}^\star(\phi) = \mathrm{ent}(\widehat{\phi}).$$

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D. Dikranjan, A. Giordano Bruno, The Bridge Theorem for totally disconnected LCA groups. Topology and its Appl. ***169*** *(2014)*

Example: the Bernoulli shifts

Let $V_c = \prod_{n=0}^{\infty} \mathbb{K}$. Clearly, V_c is linearly compact. One may define

$$ {}_{V_c}\beta \colon V_c \to V_c, \quad {}_{V_c}\beta((x_0, x_1, \dots)) = (x_1, x_2, x_3, \dots), \quad \forall (x_n)_{n\in\mathbb{N}} \in V_c, $$

which is called the *left Bernoulli shift* over V_c.

Analogously, let $V_d = \bigoplus_{n=0}^{\infty} \mathbb{K}$. Clearly, V_d is discrete. Thus

$$ \beta_{V_d} \colon V_d \to V_d, \quad \beta_V((x_0, x_1, \dots)) = (0, x_0, x_1, \dots), \quad \forall (x_n)_{n\in\mathbb{N}} \in V_d, $$

be the *right Bernoulli shift* over V_d.

Easily, one has that

$$ \boxed{V_c = \widehat{V_d} \quad \text{and} \quad {}_{V_c}\beta = \widehat{\beta_{V_d}}} $$

$$ \boxed{\mathrm{ent}(\beta_{V_d}) = 1 = \mathrm{ent}^*({}_{V_c}\beta)} $$

$$ \boxed{\mathrm{ent}({}_{V_c}\beta) = 0 = \mathrm{ent}^*(\beta_{V_d})} $$

Addition Theorem for linearly compact spaces - Part II

Theorem (C., Giordano Bruno)

Let V be an l.l.c. space, $\phi \in \mathrm{End}_{{}_{\mathbb{K}}\mathrm{LLC}}(V)$ and W a ϕ-invariant closed linear subspace of V. Then

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_W) + \mathrm{ent}(\overline{\phi})$$

if, and only if,

$$\mathrm{ent}^*(\widehat{\phi}) = \mathrm{ent}^*(\widehat{\phi} \restriction_{W^\perp}) + \mathrm{ent}^*(\overline{\widehat{\phi}}).$$

Consequently, AT holds for ent *over* ${}_{\mathbb{K}}\mathrm{LLC}$ *if, and only if, AT holds for* ent^* *over* ${}_{\mathbb{K}}\mathrm{LLC}$.

Thanks for your attention

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