Quotients of the shift map

(for frogs)

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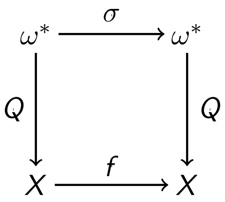
(X, f) is a *quotient* of (ω^*, σ) if there is a continuous surjection $Q : \omega^* \to X$ such that $f \circ Q = Q \circ \sigma$.



A question and two partial answers

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Theorem

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If $w(X) \leq \aleph_1$, then (X, f) is a quotient of (ω^*, σ) if and only if it is weakly incompressible.

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- For some spaces X, every continuous function $\omega^* \to X$ is induced (e.g., metric spaces).
- For other spaces this is not the case (e.g., the long line), but even for these spaces, we can come close:

Lemma (Tietze)

Suppose $X \subseteq [0,1]^\kappa$. Then every continuous map $\omega^* \to X$ is induced by a function $\omega \to [0,1]^\kappa$.

Eventual compliance

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A sequence $\langle x_n : n < \omega \rangle$ of points in X is *eventually compliant* with an open cover $\mathcal{U}$ of X provided

- each member of $\mathcal{U}$ that meets X contains a point of the sequence
- for all but finitely many n, there are $U, V \in \mathcal{U}$ such that $x_n \in U$, $x_{n+1} \in V$, and $f(U \cap X) \cap V \neq \emptyset$.

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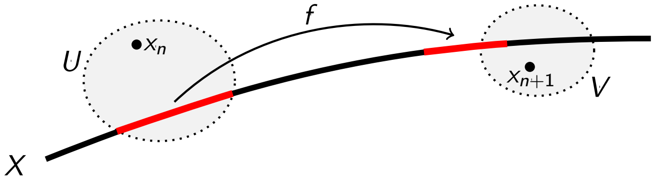

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Lemma

Let X be a closed subset of $[0,1]^\kappa$ and $f : X \to X$ continuous.

- *A sequence $\langle x_n : n < \omega \rangle$ of points in $[0,1]^\kappa$ induces a quotient mapping from (ω^*, σ) to (X, f) if and only if it is eventually compliant with every open cover.*

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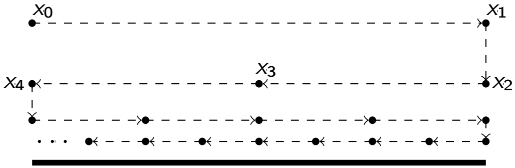
- *A sequence $\langle x_n : n < \omega \rangle$ of points in $[0,1]^\kappa$ induces a quotient mapping from (ω^*, σ) to (X, f) if and only if it is eventually compliant with every open cover.*
- *Conversely, every quotient mapping from (ω^*, σ) to (X, f) arises in this way.*

If a sequence of points is eventually compliant with every open cover, we will say it is *eventually compliant*.

Two examples

Example 1: $X = [0, 1]$ and $f = \mathrm{id}$



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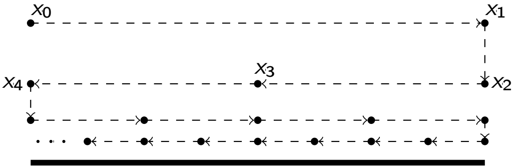
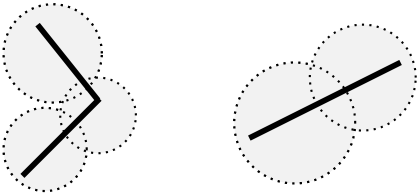


Example 2: X is disconnected and $f = \mathrm{id}$



A proof via MA(σ-centered): first attempt

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- By Bell's Theorem, $\kappa < \mathfrak{p}$ is equivalent to MA_κ(σ-centered), so it suffices to come up with a σ-centered forcing that builds the desired sequence.
- **Idea:** Let D be a countable dense subset of $[0,1]^\kappa$. A forcing condition has the form $(s, \mathcal{F})$, where s is a finite sequence of points in D and $\mathcal{F}$ is a finite collection of open covers.

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- We would like to use MA_κ(σ-centered) to get a sufficiently generic filter G of forcing conditions, and prove that $\bigcup\{s : (s, \mathcal{F}) \in G\}$ is an eventually compliant sequence.

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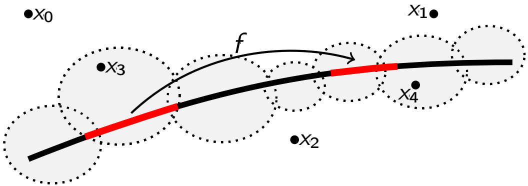

A condition where the extensions of s are restricted

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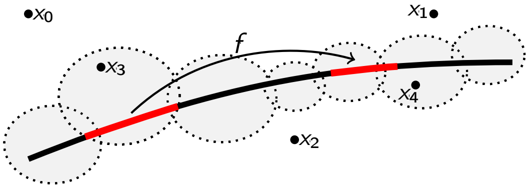

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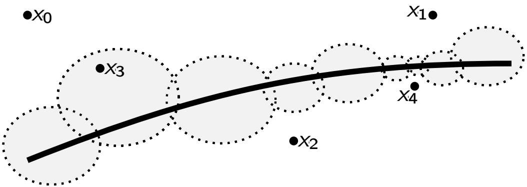

A stronger condition with no restrictions on how to extend s

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- A forcing condition is a pair $(s, \mathcal{F})$, where s is a finite sequence of points in D, $\mathcal{F}$ is a finite collection of open covers, **and the last point in s is x.**
- $(s', \mathcal{F}')$ is stronger than $(s, \mathcal{F})$ whenever $\mathcal{F}' \supseteq \mathcal{F}$ and s' extends s in a way that is compliant with every member of $\mathcal{F}$.

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This notion of forcing is σ-centered, and the generic object is a sequence of points in $[0,1]^\kappa$ that is eventually compliant with every open cover.

An inverse limit of length ω_1

Lemma

Suppose $X \subseteq [0,1]^{\omega_1}$ and $f : X \to X$ is continuous. There is a closed unbounded set of ordinals $\alpha < \omega_1$ such that for all $x, y \in X$, if $\mathrm{prj}_{[0,1]^\alpha}(x) = \mathrm{prj}_{[0,1]^\alpha}(y)$ then $\mathrm{prj}_{[0,1]^\alpha}(f(x)) = \mathrm{prj}_{[0,1]^\alpha}(f(y))$.

In other words, we may find a closed unbounded set of countable ordinals α such that projecting (X, f) onto the first α coordinates of $[0,1]^{\omega_1}$ yields a quotient mapping.

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Corollary

If (X, f) is a weakly incompressible dynamical system of weight $\aleph_1$, then it is an ω_1-length inverse limit of metrizable dynamical systems:

$$(X_0, f_0) \leftarrow (X_1, f_1) \leftarrow (X_2, f_2) \leftarrow \cdots \leftarrow (X_\alpha, f_\alpha) \leftarrow \ldots (X, f).$$

A proof strategy that, once again, almost works

1. Suppose $(X, f) = \varprojlim_{\alpha<\omega_1}(X_\alpha, f_\alpha)$, where each (X_α, f_α) is a metrizable (and weakly incompressible) dynamical system.

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3. We can try to lift this sequence through the inverse limit system: for every α, we get an eventually compliant sequence $\langle x_n^\alpha : n < \omega\rangle$ of points in (X_α, f_α), and any two of these sequences agree on coordinates where both are defined.

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4. These sequences diagonalize to give us a sequence of points in $[0, 1]^{\aleph_1}$, and this sequence will be eventually compliant with (X, f).

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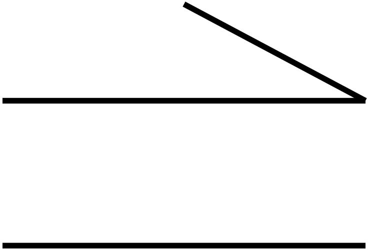
- This proof strategy is reminiscent of one of the proofs of Parovičenko's theorem (Błaszczyk and Szymański, 1980).
- But in order to accomplish step 3 of this strategy, we would need some variant of the following proposition:

Wishful thinking

Suppose $\pi_{0,1}$ is a quotient mapping from (X_1, f_1) to (X_0, f_0), and that $\langle x_n^0 : n < \omega \rangle$ is an eventually compliant sequence in (X_0, f_0). Then there is an eventually compliant sequence $\langle x_n^0 : n < \omega \rangle$ in (X_1, f_1) such that $\pi_{0,1}(x_n^1) = x_n^0$ for all n.

and this simply isn't true.

Eventually compliant sequences do not always lift through projection mappings



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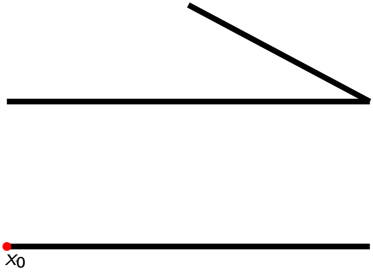

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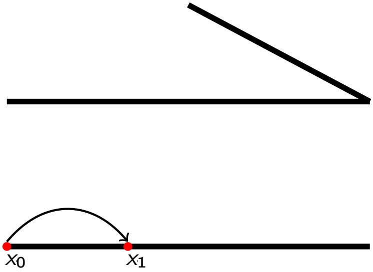

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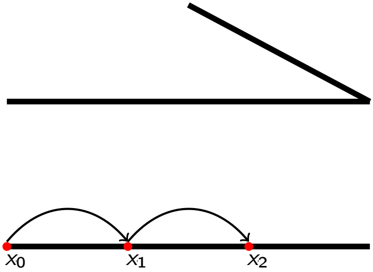

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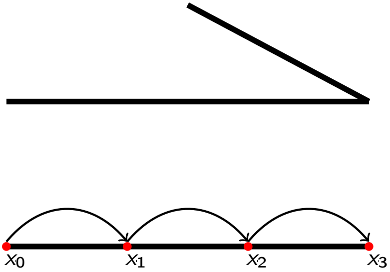

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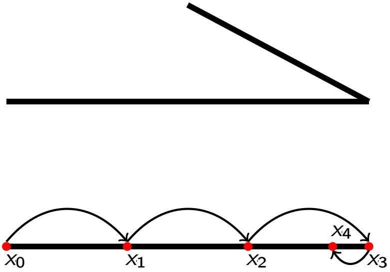

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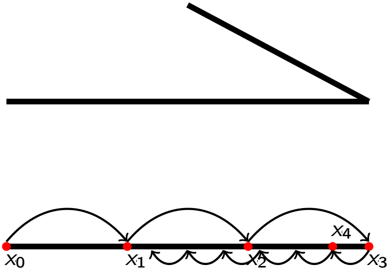

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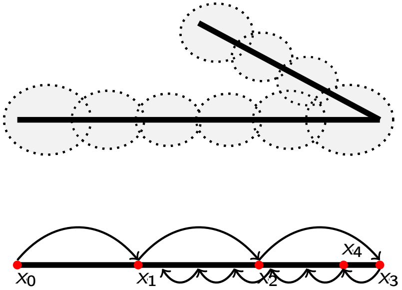

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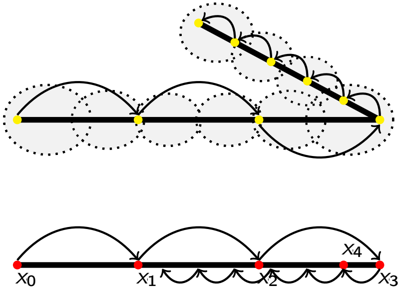

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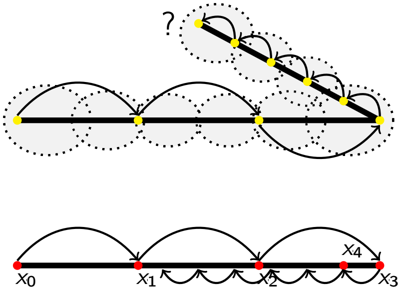

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- If a projection mapping $(X_{\alpha+1}, f_{\alpha+1}) \to (X_\alpha, f_\alpha)$ is induced by an elementary embedding, then any eventually compliant sequence in (X_α, f_α) can be lifted to $(X_{\alpha+1}, f_{\alpha+1})$.

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- If a projection mapping $(X_{\alpha+1}, f_{\alpha+1}) \to (X_\alpha, f_\alpha)$ is induced by an elementary embedding, then any eventually compliant sequence in (X_α, f_α) can be lifted to $(X_{\alpha+1}, f_{\alpha+1})$.
- This technique was pioneered by Dow and Hart to prove that every compact connected space of weight $\aleph_1$ is a continuous image of the Čech-Stone remainder of $[0, \infty)$.

Three questions

Corollary

Assuming CH, (ω^*, σ^{-1}) is a quotient of (ω^*, σ).

Question

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Theorem (Przymusiński, 1982)

Every perfectly normal compact space is a continuous image of ω^*.

Question

Suppose X is a perfectly normal compact space. Is it true that (X, f) is an abstract omega-limit set if and only if it is weakly incompressible?

The end

Thank you for listening