Strong Shape and Homology of Continuous Maps

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Abstract

In paper [Ba] the fiber resolution and fiber expansion of continuous maps are defined and it is shown that any fiber resolution is a fiber expansion. In this paper we have defined a strong fiber expansion. We have shown that any fiber resolution is a strong fiber expansion. Besides, we have proved an analogous lemma of the Main Lemma about strong expansion [Ma]. Using the obtained results and methods of strong shape theory [Ma] we have constructed a strong fiber shape category of maps of compact metric spaces [Be-Ba].

Abstract

In the second part of this paper, we have constructed the strong homological functor from the strong shape category of maps of compact metric spaces to the category of sequences of abelian groups and level mophisms. Using the obtained results we have defined the homological functor $\mathbf{H} : Mor_{CM} \to \mathscr{A}b$ from the category of continuous maps of compact metric spaces to the category of abelian groups [Be-Ba], which is strong shape invariant and has the semi-continuous property [Be]. Besides, we give an example of a map $f : X \to Y$ with trivial spectral homology and non-trivial strong homology groups [Be].

Classification of Topological Spaces

- Homotopy Classification

$$\mathbf{H}(\mathrm{Top}^2) \text{ -- Homotopy category}$$

- Strong shape classification

$$\mathbf{SSh}(\mathrm{Top}^2) \text{ -- Strong shape category}$$

- Shape Classification

$$\mathbf{Sh}(\mathrm{Top}^2) \text{ -- Shape category}$$

Corresponding Homology Theories

- Singular Homology Theory

$$\mathrm{H}_* : \mathrm{Top}^2 \to \mathscr{A}\mathrm{b}$$

- Strong (Steenrod, Total) Homology Theory

$$\mathrm{H}_*^{st} : \mathrm{Top}^2 \to \mathscr{A}\mathrm{b}$$

- Čech Homology Theory

$$\check{\mathrm{H}}_* : \mathrm{Top}^2 \to \mathscr{A}\mathrm{b}$$

Advantage and Disadvantage

Singular homology theory

- It is homotopy invariant
- AD: It is an exact homology theory
- DIS: It is not continuous homology theory
- It satisfies only the week excision axiom

Advantage and Disadvantage

Čech homology theory

- AD: It is shape invariant
- DIS: It is not exact homology theory
- AD: It is a continuous homology theory
- AD: It satisfies the excision axiom

Advantage and Disadvantage

Strong (Steenrod, Total) homology theory

- AD: It is strong shape invariant
- AD: It is an exact homology theory
- It is a semi-continuous homology theory
- AD: It satisfies the strong excision axiom

Tendency in Shape Theory

- Fiber shape theory (shape theory of continuous maps)

$$\mathbf{Sh}(\mathrm{Mor}_{\mathrm{Top}})$$

- Strong fiber shape theory (strong shape theory of continuous maps) theory

$$\mathbf{SSh}(\mathrm{Mor}_{\mathrm{Top}})$$

Fiber Shape Theory

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Theorem (V.Baladze, 1986)

For each continuous map $f : X \to Y$ of compact spaces there exist inverse systems $\mathbf{X} = \{X_\alpha, p_{(\alpha,\alpha')}, \mathbb{A}\}$ and $\mathbf{Y} = \{Y_\alpha, q_{(\alpha,\alpha')}, \mathbb{A}\}$ of compact ANR-spaces and a system $\mathbf{f} = \{f_\alpha, id\} : X \to Y$, where for each $\alpha \in \mathbb{A}$, $f_\alpha : X_\alpha \to Y_\alpha$ is mapping such that $f_\alpha p_{(\alpha,\alpha')} = q_{\alpha,\alpha'} f_{(\alpha')}$ and following are fulfilled:
1) $X \approx Lim\mathbf{X}, Y \approx Lim\mathbf{Y}, f \approx Lim\mathbf{f}$;
2) $q_\alpha f = f_\alpha p_\alpha$, where $\mathbf{p} = \{p_\alpha\} : X \to \mathbf{X}$ and $\mathbf{q} = \{q_\alpha\} : Y \to \mathbf{Y}$ are inverse limit of inverse systems $\mathbf{X} = \{X_\alpha, p_{(\alpha,\alpha')}, \mathbb{A}\}$ and $\mathbf{Y} = \{Y_\alpha, q_{(\alpha,\alpha')}, \mathbb{A}\}$.

Classification of Continuous maps

- Homotopy Classification of continuous maps

$$\mathbf{H}(\mathrm{Mor}_{\mathrm{Top}})$$

- Strong shape classification of continuous maps

$$\mathbf{SSh}(\mathrm{Mor}_{\mathrm{Top}})$$

- Shape Classification of continuous maps

$$\mathbf{Sh}(\mathrm{Mor}_{\mathrm{Top}})$$

Homology Theories of Continuous Maps

- Singular Homology Theory of Continuous maps

$$\mathrm{H}_* : \mathrm{Mor}_{\mathrm{Top}} \to \mathscr{A}\mathrm{b}$$

- Strong (Steenrod, Total) Homology Theory of Continuous maps

$$\mathrm{H}_*^{st} : \mathrm{Mor}_{\mathrm{Top}} \to \mathscr{A}\mathrm{b}$$

- Čech Homology Theory of Continuous maps

$$\check{\mathrm{H}}_* : \mathrm{Mor}_{\mathrm{Top}} \to \mathscr{A}\mathrm{b}$$

Axioms of the Strong (Steenrod, Total) Homology Theory of Continuous Maps

- It should be homotopy invariant
- It should be an exact homology theory
- It should have the semi-continuous property
- There is problem in formulating the excision axiom

Axioms of the Strong (Steenrod, Total) Homology Theory of Continuous Maps

Theorem (Mrozik):

A homology theory $\{\mathrm{H}_*, \delta\}$ on the category $\mathbf{CM^2}$ is strong shape invariant if and only if it satifies the strong excision axiom (SE).
SE: for each compact metric pair (X, A) with $A \neq \emptyset$, the quotiont map

$$\mathrm{p} : (\mathrm{X}, \mathrm{A}) \to (\mathrm{X}/\mathrm{A}, *)$$

induces an isomorphism:

$$\mathrm{p}_* : \mathrm{H}_*(\mathrm{X}, \mathrm{A}) \to \mathrm{H}_*(\mathrm{X}/\mathrm{A}, *)$$

Axioms of the Strong (Steenrod, Total) Homology Theory of Continuous Maps

- It should be strong shape invariant
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- It should have the semi-continuous property

Definition of Strong Homology Groups of Continuous Maps

- $f : X \to Y$ - A continuous map of compact mertic space

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- $f : X \to Y$ - A continuous map of compact mertic space
- $F = S(f) : X \to Y$ - The strong shape map induced by f

The strong shape map induced by $f : X \to Y$

$F = \{\mathbf{p}, \mathbf{q}, [\mathbf{f}]\}$, where $\mathbf{p} : \mathrm{X} \to \mathbf{X}$ and $\mathbf{q} : \mathrm{Y} \to \mathbf{Y}$ are strong expansions and $[\mathbf{f}] : [\mathbf{X}] \to [\mathbf{Y}]$ is coherent class of coherent mapping

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$$\begin{array}{ccc} X & \xrightarrow{\mathbf{p}} & \mathbf{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle [\mathbf{f}]} \\ Y & \xrightarrow[\mathbf{q}]{} & \mathbf{Y} \end{array}$$

Coherent Mapping of an Inverse Sequence

Definition:

Let $\mathbf{X} = \{X_n, p_{n,n+1}, \mathbb{N}\}$ and $\mathbf{Y} = \{Y_m, q_{m,m+1}, \mathbb{N}\}$ be inverse sequences in the category Top. A coherent mapping $\mathbf{f} = \{f_m, f_{m,m+1}, f\} : \mathbf{X} \to \mathbf{Y}$ consists of an increasing function $f : \mathbb{N} \to \mathbb{N}$, of maps $f_m : X_{f(m)} \to Y_m$ and $f_{m,m+1} : X_{f(m+1)} \times I \to Y_m$ such that

$$f_{m,m+1}(x, 0) = f_m(p_{f(m),f(m+1)}(x)),$$

$$f_{m,m+1}(x, 1) = q_{m,m+1} f_m(x)).$$

Coherent Mapping of an Inverse Sequence

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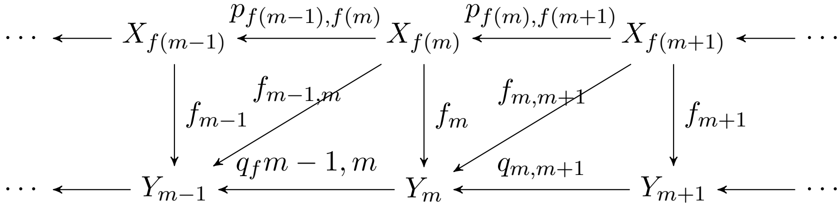


Coherent Homotopy of Coherent mapping

Definition:

Two coherent mapping $\mathbf{f}, \mathbf{f}' : X \to Y$ are coherent homotopic, $\mathbf{f} \cong \mathbf{f}'$, if there exists a coherent mapping $\mathbf{F} : \mathbf{X} \times I \to \mathbf{Y}$ such that for every $m \in \mathbb{N}$ admits an index $F(m) \geqslant f(m), f'(m)$ and following is fulfilled

$$F_m(x, 0) = f_m(p_{f(m),F(m)}(x)),$$

$$F_m(x, 1) = f'_m(p_{f'(m),F(m)}(x)).$$

Moreover

$$F_{m,m+1}(x, 0, t) = f_{m,m+1}(p_{f(m+1),F(m+1)}(x), t),$$

$$F_{m,m+1}(x, 1, t) = f'_{m,m+1}(p_{f'(m+1),F(m+1)}(x), t)$$

$$F_{m,m+1}(x, s, 0) = F_m(p_{F(m),F(m+1)}(x), s),$$

$$F_{m,m+1}(x, s, 1) = (q_{F(m),F(m+1)}F_{m+1}(x), s),$$

Definition of Strong Homology Groups of Continuous Maps

- $[\mathbf{f}] : [\mathbf{X}] \to [\mathbf{Y}]$ - coherent class of coherent mapping
- $\mathbf{f} = \{f_m, f_{m,m+1}, f\} : \mathbf{X} \to \mathbf{Y}$ - coherent mapping
- $\mathbf{F}_{\#} = \{\mathbf{f}^{\mathbf{0}}_{\#}, \mathbf{f}^{\mathbf{0}}_{\#}, \mathbf{f}^{\mathbf{1}}_{\#}\} : \mathbf{X} \to \mathbf{Y}$ - coherent mapping of chain maps

Definition of Coherent mopping of chain maps

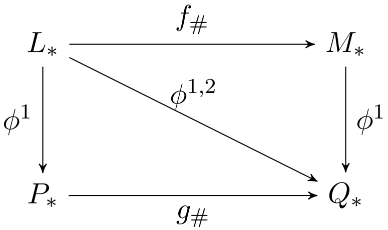
What is a coherent mapping of chain maps?

Definition:

A coherent morphism $\Phi : f_{\#} \to g_{\#}$ of chain maps is a system $\Phi = \{(\phi^1, \phi^2), \phi^{1,2}\}$, where $\phi^1 : L_* \to P_*$ and $\phi^2 : M_* \to Q_*$ are chain maps and $\phi^{1,2} : L_* \to Q_*$ is a chain homotopy of the chain maps $g_{\#}\phi^1$ and $\phi^2 f_{\#}$;

Coherent Mapping of Chain Maps

Coherent mapping of chain maps:



Definition of Coherent mopping of chain maps

Definition:

A coherent homotopy $D = \{(D^1, D^2), D^{1,2}\}$ of coherent morphisms $\Phi = \{(\phi^1, \phi^2), \phi^{1,2}\}$ and $\Psi = \{(\psi^1, \psi^2), \psi^{1,2}\}$ is a system $D = \{(D^1, D^2), D^{1,2}\}$, where D^1 is a chain homotopy of ϕ^1 and ψ^1, D^2 is also a chain homotopy of ϕ^2 and ψ^2 and $D^{1,2} : L_* \to Q_*$ is a chain map of degree two, which satisfies the following conditions:

$$\partial D^{1,2} - D^{1,2}\partial = g_\# D^1 - D^2 f_\# + \phi^{1,2} - \psi^{1,2};$$

Definition of Strong Homology Groups of Continuous Maps

- $[\mathbf{f}] : [\mathbf{X}] \to [\mathbf{Y}]$ - coherent class of coherent mapping
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- $\mathbf{F}_\# = \{\mathbf{f}^{\mathbf{0}}_\#, \mathbf{f}^{\mathbf{0}}_\#, \mathbf{f}^{\mathbf{1}}_\#\} : \mathbf{p}_\# \to \mathbf{q}_\#$ - coherent mapping of chain maps

The Chain Map Induced by a Coherent mapping

Lemma

Each coherent mapping $\mathbf{f} = \{f_m, f_{m,m+1}, f\} : \mathbf{X} \to \mathbf{Y}$ induces coherent mapping of chain maps

$$\mathbf{F}_{\#} = \{\mathbf{f}^{\mathbf{0}}_{\#}, \mathbf{f}^{\mathbf{0}}_{\#}, \mathbf{f}^{\mathbf{1}}_{\#}\} : \mathbf{p}_{\#} \to \mathbf{q}_{\#}$$

Lemma

Coherent homotopic coherent mappings

$$\mathbf{f} \simeq \mathbf{f}' : \mathbf{X} \to \mathbf{Y}$$

induce coherent mappings of chain maps

$$\mathbf{F}_{\#} \simeq \mathbf{F}'_{\#} : \mathbf{p}_{\#} \to \mathbf{q}_{\#}$$

The Chain Map Induced by a Coherent mapping

Lemma

A coherent mapping of chain maps $\Phi : f_\# \to g_\#$ induces the chain map

$$\Phi_\# : C_*(f_\#) \to C_*(g_\#),$$

where $C_*(-)$ is chain cone of chain map.

Lemma

Coherent homotopic coherent mappings of chain maps $D : \Phi \simeq \Psi$ induce the homotopic chain maps

$$\Phi_\# \simeq \Psi_\# : C_*(f_\#) \to C_*(g_\#).$$

The Chain Map Induced by a Coherent mapping

Theorem

Each coherent mapping $\mathbf{f} = \{f_m, f_{m,m+1}, f\} : \mathbf{X} \to \mathbf{Y}$ induces chain map

$$\mathbf{F}_{\#} : C_*(\mathbf{p}_{\#}) \to C_*(\mathbf{q}_{\#}),$$

Theorem

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induce homotopic chain maps

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Definition of Strong Homology Groups of Continuous Maps

Definition

Given a continuous map $f : X \to Y$ of compact metric spaces, the $(n+1)$-dimension homology groups of chain cone $C_*(\mathbf{F}_\#)$ of chain map $\mathbf{F}_\# : C_*(\mathbf{p}_\#) \to C_*(\mathbf{q}_\#)$ is called n-dimension strong homology groups of f.

$$\mathbf{H_n}(f) \approx \mathrm{H}_{\mathrm{n+1}}(C_*(\mathbf{F}_\#)).$$

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Theorem

For each inclusion $i : A \to X$ of compact metric spaces, there exists the isomorphism

$$\mathbf{H}_*(i) \approx \mathbf{H}_*(X, A).$$

Strong Shape Invariant

Theorem:

If continuous maps $f, g : X \to Y$ of compact metric spaces induces the same strong shape mapping then their strong homology groups are isomorphic

$$\mathbf{H_n}(f) \approx \mathbf{H_n}(g)$$

Exactness Property

Theorem

For each continuous map $f : X \to Y$ of compact metric spaces there exists a long exact sequence

$$\cdots \longrightarrow \mathbf{H}_n(X) \longrightarrow \mathbf{H}_n(Y) \longrightarrow \mathbf{H}_n(f) \to \mathbf{H}_{n-1}(X) \longrightarrow \cdots$$

Semi-continuity

Theorem

For each continuous map $f : X \to Y$ of compact metric spaces there exists a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 H_{n+1}(f_i) \longrightarrow \mathbf{H}_n(f) \longrightarrow \varprojlim H_n(f_i) \longrightarrow 0$$

where $\mathbf{f} = \{f_i\} : \mathbf{X} \to \mathbf{Y}$ is a mapping of inverse sequences of compact metric ANR-spaces such that
1) $X = \varprojlim \mathbf{X}$, $Y = \varprojlim \mathbf{Y}$, $f = \mathbf{f}$;
2)$\{q_i\}f = \{f_i\}\{p_i\}$ and $\mathbf{p} = \{p_i\} : X \to \mathbf{X}$ and $\mathbf{q} = \{q_i\} : Y \to \mathbf{Y}$ are inverse limits of the inverse sequences $\mathbf{X}$ and $\mathbf{Y}$ (strong expansions), respectively.

Resolution of a continuous map

Definition (see [Ba]

A fiber resolution of a map f is a morphism $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p_\lambda^{'})\} : f \to \mathbf{f}$ of the category $\mathbf{pro} - Mor_{Top}$ which, for any ANR-map $t : P \to P'$ and a pair (α, α') of coverings $\alpha \in Cov(P)$ and $\alpha' \in Cov(P')$, satisfies the following two conditions:

FR1) for every morphism $(\varphi, \varphi') : f \to t$ there exist $\lambda \in \Lambda$ and a morphism $(\varphi_\lambda, \varphi_\lambda^{'}) : f_\lambda \to t$ such that $(\varphi_\lambda, \varphi_\lambda^{'}) \cdot (p_\lambda, p_\lambda^{'})$ and (φ, φ') are (α, α')-near;

FR2) there exists a pair (β, β') of coverings $\beta \in Cov(P)$ and $\beta' \in Cov(P')$ with the following property: if $\lambda \in \Lambda$ and $(\varphi_\lambda, \varphi_\lambda^{'}), (\psi_\lambda, \psi_\lambda^{'}) : f_\lambda \to t$ are morphisms such that the morphisms $(\varphi_\lambda, \varphi_\lambda^{'}) \cdot (p_\lambda, p_\lambda^{'})$ and $(\psi_\lambda, \psi_\lambda^{'}) \cdot (p_\lambda, p_\lambda^{'})$ are (β, β')-near, then there exists a $\lambda' \geq \lambda$ such that $(\varphi_\lambda, \varphi_\lambda^{'}) \cdot (p_{\lambda\lambda'}, p_{\lambda\lambda'}^{'})$ and $(\psi_\lambda, \psi_\lambda^{'}) \cdot (p_{\lambda\lambda'}, p_{\lambda\lambda'}^{'})$ are (α, α')-near.

Strong expansion of a continuous map

Definition (cf. [Ma])

We will say that a morphism $(\mathbf{p}, \mathbf{p'}) = \{(p_\lambda, p'_\lambda)\} : f \to \mathbf{f}$ of the category $\mathbf{pro} - Mor_{Top}$ is strong fiber expansion of a continuous map $f : X \to X'$ if for every ANR-map $t : P \to P'$ the following conditions are fulfilled:

SF1) for every morphism $(\varphi, \varphi') : f \to t$ there exist $\lambda \in \Lambda$ and a morphism $(\varphi_\lambda, \varphi'_\lambda) : f_\lambda \to t$ such that $(\varphi, \varphi') \cong (\varphi_\lambda, \varphi'_\lambda) \cdot (p_\lambda, p'_\lambda)$

FR2) if $\lambda \in \Lambda$ and $(\varphi_\lambda, \varphi'_\lambda), (\psi_\lambda, \psi'_\lambda) : f_\lambda \to t$ are morphisms such that the morphisms $(\varphi_\lambda, \varphi'_\lambda) \cdot (p_\lambda, p'_\lambda)$ and $(\psi_\lambda, \psi'_\lambda) \cdot (p_\lambda, p'_\lambda)$ are connected by a fiber homotopy $(\Theta, \Theta') : f \times 1_I \to t$, then there exists a $\lambda' \geq \lambda$ and fiber homotopies $(\Delta, \Delta') : f_\lambda \times 1_I \to t$ and $(\Gamma, \Gamma') : f \times 1_I \times 1_I \to t$ such that the homotopy (Δ, Δ') connects the morphisms $(\varphi_\lambda, \varphi'_\lambda) \cdot (p_{\lambda\lambda'}, p'_{\lambda\lambda'})$ and $(\psi_\lambda, \psi'_\lambda) \cdot (p_{\lambda\lambda'}, p'_{\lambda\lambda'})$ and the homotopy (Γ, Γ') connects (Θ, Θ') and $(\Delta, \Delta') \cdot (p_\lambda \times 1_I, p'_\lambda \times 1_I)$ and is fixed on the submap $f \times 1_{\partial I} : X \times 1_{\partial I} \to X' \times 1_{\partial I}$.

Coherent expansion of a continuous map

Definition (cf. [Ma])

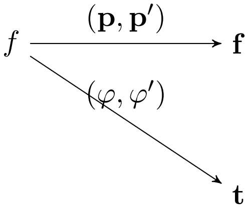
A fiber coherent expansion of a continuous map $f : X \to X'$ is a morphism $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p_\lambda^{'})\} : f \to \mathbf{f}$ of the category CH($\mathbf{pro} - Mor_{Top}$), which has the following property:
CH) for every system $\mathbf{t}$ in the category CH($\mathbf{pro} - Mor_{ANR}$) and every morphisms $(\varphi, \varphi') : f \to \mathbf{t}$ of CH($\mathbf{pro} - Mor_{Top}$), there exists a unique morphism $[(\varphi, \varphi')] : [\mathbf{f}] \to \mathbf{t}$ of the category CH($\mathbf{pro} - Mor_{Top}$) such that

$$(\varphi, \varphi') = (\mathbf{p}, \mathbf{p}')[(\varphi, \varphi')]$$

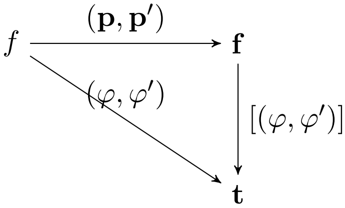
Coherent expansion of a continuous map

$$f \xrightarrow{(\mathbf{p}, \mathbf{p}')} \mathbf{f}$$

Coherent expansion of a continuous map



Coherent expansion of a continuous map



Construction of strong shape theory of maps

Lemma (cf. [Ba])

Let $f : X \to X'$ be a map of topological spaces, $t' : P_1 \to P_1^{'}$, $t : P \to P'$ be ANR-maps. If $(\zeta, \zeta') : f \to t'$, (ξ, ξ'), $(\eta, \eta') : t' \to t$ are morphisms and $(\Theta, \Theta') : f \times 1_I \to t$ is homotopy which connects the morphisms $(\xi, \xi') \cdot (\zeta, \zeta')$ and $(\eta, \eta') \cdot (\zeta, \zeta')$, then there exists an ANR-map $t'' : P_2 \to P_2^{'}$, a morphism $(\sigma, \sigma') : f \to t''$, $(\kappa, \kappa') : t'' \to t'$ and a fiber homotopy $(\Delta, \Delta') : t'' \times 1_I \to t$ such that

$$
\begin{gathered}
(\zeta, \zeta') = (\kappa, \kappa') \cdot (\sigma, \sigma'), \\
(\Delta, \Delta') : (\xi, \xi') \cdot (\kappa, \kappa') \cong (\eta, \eta') \cdot (\kappa, \kappa'), \\
(\Theta, \Theta') = (\Delta, \Delta') \cdot (\sigma \times 1_I, \sigma_1^{'} \times 1_I).
\end{gathered}
$$

Construction of strong shape theory of maps

Lemma (cf. [Ma])

Let $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p_\lambda^{'})\} : f \to \mathbf{f}$ be a fiber resolution and let $\lambda \in \Lambda$, $t : P \to P'$ be a continuous map, $(\varphi_\lambda, \varphi_\lambda^{'}), (\psi_\lambda, \psi_\lambda^{'}) : f_\lambda \to t$ be a morphism and $(\Theta, \Theta') : f \times 1_I \to t$ be as in SF2). Then for every pair (α, α') of coverings $\alpha \in Cov(P)$ and $\alpha' \in Cov(P')$, there exist $\lambda' \geq \lambda$ and a fiber homotopy $(\Delta, \Delta') : f_{\lambda'} \times 1_I \to t$ such that

$$(\Delta, \Delta') : (\varphi_\lambda, \varphi_\lambda^{'}) \cdot (p_{\lambda\lambda'}, p_{\lambda\lambda'}^{'}) \cong (\psi_\lambda, \psi_\lambda^{'}) \cdot (p_{\lambda\lambda'}, p_{\lambda\lambda'}^{'}),$$
$$((\Theta, \Theta'), (\Delta, \Delta') \cdot (p_\lambda \times 1_I, p_\lambda^{'} \times 1_I)) \leq (\alpha, \alpha').$$

Construction of strong shape theory of maps

Lemma (cf. [Ma])

Every fiber resolution $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p_\lambda^{'})\} : f \to \mathbf{f}$ of a map $f : X \to X'$ is a strong fiber expansion of f.

Construction of strong shape theory of maps

Lemma (cf. Main Lemma [Ma])

For every strong fiber expansion $(\mathbf{p}, \mathbf{p}') : f \to \mathbf{f}$, if $\lambda \in \Lambda$, $(\sigma_\lambda, \sigma_\lambda^{'}) : f_\lambda \times 1_{\partial I^2} \to t$ is a morphism and $(\Theta, \Theta') : f \times 1_{I^2} \to t$ is a fiber homotopy such that

$$(\Theta_{|X\times\partial I^2}, \Theta^{'}_{|X'\times\partial I^2}) = (\Theta, \Theta')\cdot(i_{X\times\partial I^2}, i_{X'\times\partial I^2}) = (\sigma_\lambda, \sigma_\lambda^{'})\cdot(p_\lambda \times 1_{\partial I^2}, p_\lambda^{'} \times 1_{\partial I^2})\ ,$$

then there exists a $\lambda' \geq \lambda$ and fiber homotopies $(\Delta, \Delta') : f_\lambda \times 1_{I^2} \to t$ and $(\Gamma, \Gamma') : f_\lambda \times 1_{I^2} \times 1_I \to t$ such that

$$(\Delta_{|X_{\lambda'}\times\partial I^2}, \Delta^{'}_{|X^{'}_{\lambda'}\times\partial I^2}) = (\Delta, \Delta')\cdot(i_{X_{\lambda'}\times\partial I^2}, i_{X^{'}_{\lambda'}\times\partial I^2}) = (\sigma_\lambda, \sigma_\lambda^{'})\cdot(p_{\lambda\lambda'} \times 1_{\partial I}, p_{\lambda\lambda'}^{'} \times 1_{\partial I})$$

and the homotopy (Γ, Γ') connects (Θ, Θ') and $(\Delta, \Delta')\cdot(p_\lambda \times 1_{I^2}, p_\lambda^{'} \times 1_{I^2})$ and is fixed on the submap

Construction of strong shape theory of maps

Lemma (cf. [Ma])

Every strong fiber expansion $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p'_\lambda)\} : f \to \mathbf{f}$ of a map $f : X \to X'$ is a coherent fiber expansion of f.

Construction of strong shape theory of maps

Theorem

If $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p_\lambda^{'})\} : f \to \mathbf{f}$ is a strong fiber ANR-expansion of a continuous map $f : X \to X'$ of compact metric spaces, then for each coherent morphism $(\varphi, \varphi') : f \to \mathbf{g}$, where $\mathbf{g} \in \mathrm{CH}(\mathbf{tow} - \mathrm{Mor}_{\mathrm{ANR}})$ there exists a coherent morphism $[(\psi, \psi')] : \mathbf{f} \to \mathbf{g}$ such that (φ, φ') is coherent homotopic to $(\psi, \psi')[(\mathbf{p}, \mathbf{p}')]$.

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Theorem

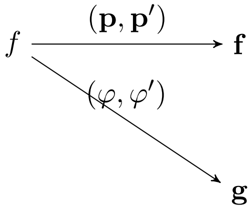
If $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p^{'}_\lambda)\} : f \to \mathbf{f}$ is a strong fiber ANR-expansion of a continuous map $f : X \to X'$ of compact metric spaces, then for each coherent morphism $(\varphi, \varphi') : f \to \mathbf{g}$, where $\mathbf{g} \in \mathrm{CH}(\mathbf{tow} - \mathrm{Mor}_{\mathrm{ANR}})$ there exists a coherent morphism $[(\psi, \psi')] : \mathbf{f} \to \mathbf{g}$ such that (φ, φ') is coherent homotopic to $(\psi, \psi')[(\mathbf{p}, \mathbf{p}')]$.

$$f \xrightarrow{\quad (\mathbf{p}, \mathbf{p}') \quad} \mathbf{f}$$

Construction of strong shape theory of maps

Theorem

If $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p_\lambda^{'})\} : f \to \mathbf{f}$ is a strong fiber ANR-expansion of a continuous map $f : X \to X'$ of compact metric spaces, then for each coherent morphism $(\varphi, \varphi') : f \to \mathbf{g}$, where $\mathbf{g} \in \mathrm{CH}(\mathbf{tow} - \mathrm{Mor}_{\mathrm{ANR}})$ there exists a coherent morphism $[(\psi, \psi')] : \mathbf{f} \to \mathbf{g}$ such that (φ, φ') is coherent homotopic to $(\psi, \psi')[(\mathbf{p}, \mathbf{p}')]$.



Construction of strong shape theory of maps

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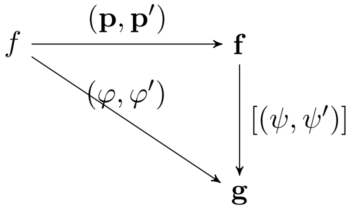

Construction of strong shape theory of maps

Theorem

If $(\mathbf{p}, \mathbf{p}') = \{(p_\lambda, p_\lambda^{'})\} : f \to \mathbf{f}$ is a strong fiber ANR-expansion of a continuous map $f : X \to X'$ of compact metric spaces, $(\varphi, \varphi') : f \to \mathbf{g}$ is coherent morphisms, where $\mathbf{g} \in \mathbf{CH}(\mathbf{tow} \cdot \mathbf{Mor_{CM}})$ and $(\psi_1, \psi_1^{'}), (\psi_2, \psi_2^{'}) : \mathbf{f} \to \mathbf{g}$ are coherent morphisms such that that (φ, φ') is coherent homotopic to $(\psi_1, \psi_1^{'}) \cdot (\mathbf{p}, \mathbf{p}')$ and $(\psi_2, \psi_2^{'}) \cdot (\mathbf{p}, \mathbf{p}')$, then the morphisms $(\psi_1, \psi_1^{'})$ and $(\psi_2, \psi_2^{'})$ coherent homotopic.

Construction of strong shape theory of maps

For each pair $f, g \in \mathbf{Mor_{CM}}$ consider the set of all triples

$$((\mathbf{p}, p'), (\mathbf{q}, q'), [(\psi, \psi')]),$$

where $(\mathbf{p}, \mathbf{p}') : f \to \mathbf{f}$ and $(\mathbf{q}, \mathbf{q}') : g \to \mathbf{g}$ are strong fiber expansion and $[(\psi, \psi')]$ is a coherent homotopy class of the coherent morphism $(\psi, \psi') : \mathbf{f} \to \mathbf{g}$.

Construction of strong shape theory of maps

Definition

The equivalence class of the triple $((\mathbf{p}, p'), (\mathbf{q}, q'), [(\psi, \psi')])$ is denoted by $F : f \to g$ and is called a strong shape morphism from f to g. Let

$$\mathbf{SSh}(Mor_{CM})$$

be the category of all continuous maps of compact metric spaces and all strong shape morphisms, called the strong fiber shape category of Mor_{CM}.

Strong homology theory of maps

Theorem

There exists the homological functor

$$H : \mathbf{SSh}(Mor_{CM}) \to \mathscr{A}b$$

such that for each continuous map $f : X \to Y$ of compact metric spaces

$$\mathbf{H_n}(f) \approx H_n(F(f)),$$

where $\mathbf{H_n}(f)$ is strong homology group of f and $F(f) : X \to Y$ is strong shape morphism induced by $f : X \to Y$.

Strong homology theory of maps

Let

$$F : Mor_{CM} \to \mathbf{SSh}(Mor_{CM})$$

be strong shape functor. Define strong homology functor on the category Mor_{CM} as a composition of the functors F and H and denote it by $\mathbf{H}$. i.e.

$$\mathbf{H} \equiv (H \circ F) : Mor_{CM} \to \mathscr{A}b$$

Strong homology theory of maps

For the strong homology functor

$$\mathbf{H} : Mor_{CM} \to \mathscr{A}b$$

the following is fulfilled:

Theorem

For each continuous map $f : X \to Y$ of compact metric spaces there exists a long exact sequence

$$\cdots \longrightarrow \mathbf{H}_n(X) \longrightarrow \mathbf{H}_n(Y) \longrightarrow \mathbf{H}_n(f) \to \mathbf{H}_{n-1}(X) \longrightarrow \cdots$$

Strong homology theory of maps

Theorem

Any morhism $(\phi, \psi) : f \to f'$ from the category Mor_{Top} induces morphism $(\phi, \psi)_* : \mathbf{H}_*(f) \to \mathbf{H}_*(f')$ of exact sequences:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathbf{H}_n(X) & \longrightarrow & \mathbf{H}_n(Y) & \longrightarrow & \mathbf{H}_n(f) & \longrightarrow & \mathbf{H}_{n-1}(X) & \longrightarrow & \cdots \\
 & & \downarrow \phi_* & & \downarrow \psi* & & \downarrow (\phi,\psi)_* & & \downarrow \phi_* & & \\
\cdots & \longrightarrow & \mathbf{H}_n(X') & \longrightarrow & \mathbf{H}_n(Y') & \longrightarrow & \mathbf{H}_n(f') & \longrightarrow & \mathbf{H}_{n-1}(X') & \longrightarrow & \cdots
\end{array}$$

Strong homology theory of maps

Theorem

If two morhisms $(\phi_1, \psi_1), (\phi_2, \psi_2) : f \to f'$ have a same strong shape then they induce the same morphisms between of coresponding homological sequences:

$$(\phi_1, \psi_1)_* = (\phi_2, \psi_2)_* : \mathbf{H}_*(f) \to \mathbf{H}_*(f')$$

Strong homology theory of maps

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Thank You!