On a problem of Ellis and Pestov's Conjecture

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We call X a G-flow.

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Y is a factor of X if there is a surjective homomorphism $X \longrightarrow Y$.

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- π extends to a continuous semigroup action $E(X) \times X \longrightarrow X$
- $\overline{Gx} = E(X)x$ for every $x \in X$.
- $(x, y) \in X^2$ is proximal iff there is $p \in E(X)$ such that $px = py$.

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Theorem (Ellis)

The universal minimal flow exists and it is unique up to isomorphism.

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$S(G)$ - compact right-topological semigroup.

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NO for groups with proximal universal minimal flows.

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If G is precompact then $S(G) = M(G) = E(M(G))$.

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Theorem (Ellis - Numakura)

Every compact right-topological semigroup S contains an idempotent, that is, $s \in S$ such that $ss = s$.

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5. All minimal left ideals are isomorphic.

Reformulation of Ellis' problem

Do homomorphisms from $S(G)$ to $M(G)$ separate points? That is, given $x \neq y \in S(G)$, is there a homomorphism $\phi : S(G) \longrightarrow M(G)$ such that $\phi(x) \neq \phi(y)$?

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Example (Glasner and Weiss, 2003)

$M(\text{Homeo}(2^\omega))$ is proximal.

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Theorem (Pestov)

$S(G) =$ *Stone space of* B, *that is, space of all ultrafilters on* B.

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Theorem (B.)

$M(G)$ is the Stone space of a maximal syndetic subalgebra of B, that is, a subalgebra of B consisting of syndetic sets and invariant under G.

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Theorem (Zucker)

There are minimal left ideals M, N in $K(S(G))$ such that idempotents in $M \cup N$ form a semigroup.

$K(S(G))$ for G – automorphism group of a countable structure

Theorem (B. + Zucker, 2016)

If $M(G)$ is metrizable then $K(S(G))$ is closed.

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In contrast, if G is discrete, then $K(\beta G)$ is never closed.

OBRIGADA!