HYPERSPACES OF EUCLIDEAN SPACES IN THE GROMOV-HAUSDORFF METRIC

SERGEY A. ANTONYAN

National University of Mexico

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The Gromov-Hausdorff distance

Definition

Let (M, d) be a metric space. For two subsets $A, B \subset M$, the Hausdorff distance $d_H(A, B)$ is defined as follows:

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

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2^M denotes the set of all nonempty compact subsets of M.

$$(2^M, d_H) \qquad \text{is a metric space.}$$

The Gromov-Hausdorff distance d_{GH} is a useful tool for studying topological properties of families of metric spaces. M. Gromov first introduced the notion of Gromov-Hausdorff distance in his ICM 1979 address in Helsinki on synthetic Riemannian geometry.

Two years later d_{GH} appeared in the book M.Gromov [3]. It turns the set GH of all isometry classes of compact metric spaces into a metric space.

For two compact metric spaces X and Y the number $d_{GH}(X, Y)$ is defined to be the infimum of all Hausdorff distances $d_H(i(X), j(Y))$ for all metric spaces M and all isometric embeddings $i : X \hookrightarrow M$ and $j : Y \hookrightarrow M$.

$$d_{GH}(X, Y) = inf\{d_H(i(X), j(Y)) \mid i : X \hookrightarrow M, \;\; j : Y \; \hookrightarrow M\}.$$

Clearly, the Gromov-Hausdorff distance between isometric spaces is zero; it is a metric on the family GH of isometry classes of compact metric spaces. The metric "space" $(\mathrm{GH}, \mathrm{d}_{\mathrm{GH}})$ is called the Gromov-Hausdorff space.

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Theorem (Huhunaishvili, 1955)

The property (3) holds true for compact isometric subsets $A \subset \mathbb{U}$, $B \subset \mathbb{U}$.

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Theorem (Berestovsky and Vershik)

The Gromov-Hausdorff distance may be computed by the following formula:

$$d_{GH}(X,Y) = \inf\{d_H\big(i(X), j(Y)\big) \mid i : X \hookrightarrow \mathbb{U},\ \ j : Y \hookrightarrow \mathbb{U}\}$$

where inf *is taken over all isometric embeddings* $i : X \hookrightarrow \mathbb{U}$ *and* $j : Y \hookrightarrow \mathbb{U}$*.*

Denote by $\mathrm{Iso}\,\mathbb{U}$ the group of all isometries of $\mathbb{U}$.

Theorem (Gromov)

$$\mathrm{GH} \cong 2^{\mathbb{U}}/\mathrm{Iso}\,\mathbb{U} \quad (\textit{an isometry}).$$

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However it is not known whether GH is an AR? Is GH$\cong \ell_2$?

In this talk we mainly are interested in the following subspaces of GH denoted by

$$\mathrm{GH}(\mathbb{R}^{\mathrm{n}}),\ \mathrm{n} \geq 1$$

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The Euclidean-Hausdorff distance

For any two compact subsets X, Y which admit an isometric embeddings in a Euclidean space $\mathbb{R}^n$, $n \geq 1$, define the Euclidean-Hausdorff distance by the following formula:

$$d_{EH}(X, Y) = \inf\{d_H\big(i(X), j(Y)\big) \mid i : X \hookrightarrow \mathbb{R}^n, \;\; j : Y \hookrightarrow \mathbb{R}^n\}$$

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Theorem (Well-known)

If X and Y are two isometric subsets of a Euiclidean space $\mathbb{R}^n$, $n \geq 1$, then there exists a Euclidean isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(X) = Y$.

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Corollary

If $X, Y \subset \mathbb{R}^n$ are compact subsets, then

$$d_{EH}(X, Y) = \inf\{d_H\big(X, F(Y)\big) \mid F : \mathbb{R}^n \to \mathbb{R}^n \ \text{ is an isometry}\}$$

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metrizes the orbit space $2^{\mathbb{R}^n}/E(n)$, and clearly,

$$\rho([X], [Y]) = d_{EH}(X, Y).$$

Main results

Clearly, $d_{GH} \leq d_{EH}$. In general $d_{GH}(X, Y)$ may be strictly less than $d_{EH}(X, Y)$.
For instance, take $X = \{a, b, c\}$ - the vertices of an equilateral triangle of side lenght 1, and $Y = \{*\}$.

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Then $d_{EH}(X, Y) = \sqrt{3}/3$ while $d_{GH}(X, Y) = 1/2$.

Theorem

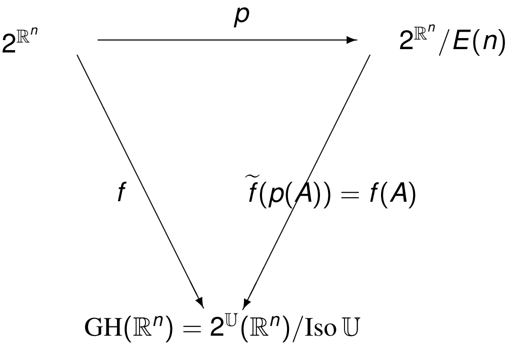
$$\mathrm{GH}(\mathbb{R}^n) \cong 2^{\mathbb{R}^n}/\mathrm{E}(n).$$

Sketch

$$2^{\mathbb{U}}(\mathbb{R}^n) = \{A \in 2^{\mathbb{U}} \mid \exists\, i : A \hookrightarrow \mathbb{R}^n\}.$$

$$f : 2^{\mathbb{R}^n} \to 2^{\mathbb{U}}(\mathbb{R}^n)/\mathrm{Iso}\,\mathbb{U} = \mathrm{GH}(\mathbb{R}^n), \qquad A \mapsto [j(A)],$$

where $j : A \hookrightarrow \mathbb{U}$ is an embedding.



Since $d_{GH} \leq d_{EH}$, we infer that

$$\widetilde{f} : 2^{\mathbb{R}^n}/E(n) \to 2^{\mathbb{U}}(\mathbb{R}^n)/\mathrm{Iso}\,\mathbb{U}$$

is continuous and bijective.

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Theorem (Memoli)

$d_{EH} \leq C_n \cdot \sqrt{d_{GH}},$

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Theorem (Memoli)

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Thus

$$\widetilde{f} : 2^{\mathbb{R}^n}/E(n) \to 2^{\mathbb{U}}(\mathbb{R}^n)/\mathrm{Iso}\,\mathbb{U}$$

is a homeomorphism:

$$GH(\mathbb{R}^n) \cong 2^{\mathbb{R}^n}/E(n).$$

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$$E(n) \curvearrowright 2^{\mathbb{R}^n}$$

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Here *proper* means that for any compact subset $K \subset 2^{\mathbb{R}^n}$, the transporter

$$\langle K, K\rangle = \{g \in E(n) \mid gK \cap K \neq \emptyset\}$$

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Facts

- In a proper G-space each stabilizer $G_x = \{g \in G \mid gx = x\}$ is **compact.**
- Every obit $G(x)$ is **closed** and $G(x) \cong_G G/G_x$,

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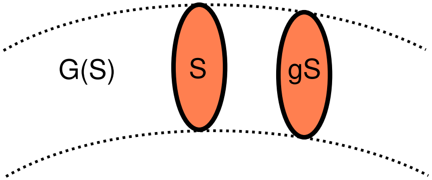

The Chebyshev balls

Theorem (P.L. Chebyshev)

For every compact subset $A \subset \mathbb{R}^n$, there is a unique closed ball $Ch(A)$, called the Chebyshev ball of A, such that $A \subset Ch(A)$.

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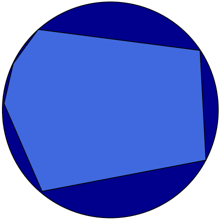

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Theorem

$$ch : 2^{\mathbb{R}^n} \to \mathbb{R}^n$$

is an $E(n)$-equivariant map, i.e.,

$$ch(gA) = gch(A), \quad A \in 2^{\mathbb{R}^n}, \quad g \in E(n).$$

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Corollary

The inverse image $T(\mathbb{R}^n) := ch^{-1}(0)$ is a global $O(n)$-slice for $2^{\mathbb{R}^n}$.

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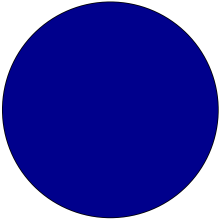
Theorem

$$\mathrm{GH}(\mathbb{R}^n) = 2^{\mathbb{R}^n}/E(n) \cong T(\mathbb{R}^n)/O(n).$$

How to compute $T(\mathbb{R}^n)/O(n)$.?

Denote

$$Ch(n) := \{A \in 2^{\mathbb{R}^n} \mid Ch(A) = \mathbb{B}^n\}.$$



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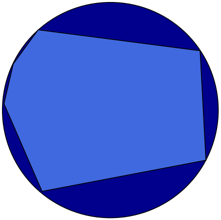

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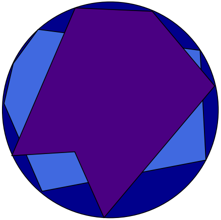

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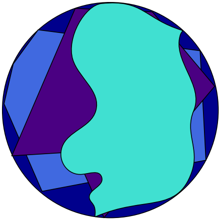

Proposition

1. $$T(\mathbb{R}^n) \cong_{O(n)} OCone\big(Ch(n)\big).$$

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Proof.

$$f(A) = \begin{cases} \frac{1}{R(A)} \cdot A, & if \;\; R(A) \neq 0 \\ \theta, & if \;\; A = \{0\}. \end{cases}$$

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But, it is well known (T.A. Chapman) that the open cone $OCone(Q) \cong Q \setminus \{*\}$.

□

[1] S.A. Antonyan, *West's problem on equivariant hyperspaces and Banach-Mazur compacta*, Trans. Amer. Math. Soc. **355, no. 8** (2003), 3379-3404.

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THE END !

Some ideas of the proof that $Ch(n)/O(n) \cong Q$.

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A a compact metrizable space X is homeomorphic to the Hilbert cube iff

- *X is an* AR.
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Theorem (S. Antonyan, 1988)

*Let G be a compact group, X a metrizable G-*AR*. Then the orbit space X/G is an* AR.

$\mathrm{DD}^n\mathrm{P}$ and DDP

Definition

Y satisfies $\mathrm{DD}^n\mathrm{P}$ for a given integer $n \geq 0$, if each map $f : \mathbb{B}^n \to Y$ can be arbitrary closely approximated by two maps $f_1, f_2 : \mathbb{B}^n \to Y$ with disjoint images.

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Proposition

A compact metric ANR space X satisfies the property DDP iff for every $\varepsilon > 0$, there exist two continuous maps $f_\varepsilon, g_\varepsilon : X \to X$ such that:

1. $\rho(x, f_\varepsilon(x)) < \varepsilon$ and $\rho(x, g_\varepsilon(x)) < \varepsilon$ for all $x \in X$.
2. $\mathrm{Im}\, f_\varepsilon \cap \mathrm{Im}\, g_\varepsilon = \emptyset$.

Equivariant DDP

Theorem

For every $\varepsilon > 0$, there exist two continuous $O(n)$-equivariant maps $f_\varepsilon, g_\varepsilon : Ch(n) \to Ch(n)$ such that:

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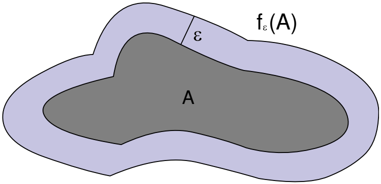

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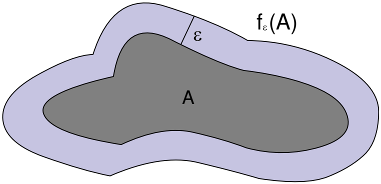

It is clear that $f_\varepsilon(A) \in Ch(n)$ whenever $A \in Ch(n)$.

Proof.

• The second map

$$g_\varepsilon : Ch(n) \to Ch(n)$$

is defined in such a way that

$$\operatorname{Int} g_\varepsilon(A) = \emptyset.$$

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