

Some results and problems on Countable Dense Homogeneous spaces

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All spaces under discussion are separable and metrizable. A topological space X is Countable Dense Homogeneous (abbreviated: CDH) provided that for all countable dense subsets D and E of X there is a homeomorphism $f : X \rightarrow X$ such that $f(D) = E$. Well-known examples of CDH-spaces include the real line, the Cantor set and the Hilbert cube. We discuss some old and some new problems on CDH-ness. An old and very tough problem is whether every connected Polish CDH-space is locally connected. This is known to be true for locally compact spaces. We provide a partial answer to the problem which has the result for locally compact spaces as a corollary. The old problem whether there is a meager CDH-space was solved by Farah, Hrušák, and Martínez Ranero in ([1]). Their example has size \aleph_1 . A relatively new result by Hernández-Gutiérrez, Hrušák and van Mill ([2]) is that such spaces exist for every cardinality less than or equal to \mathfrak{b} . Such spaces are necessarily λ -sets. That are spaces in which every countable subspace is G_δ . We prove, in joint work with Hrušák, that the existence of a nontrivial connected meager CDH-space is independent of ZFC. The main new result here is that under the Continuum Hypothesis, there is a connected dense CDH-subspace of the Hilbert cube which is a λ -set. Whether such a set exists in the plane, remains an intriguing open problem.

- [1] I. Farah, M. Hrušák, and C. M. Ranero, *A countable dense homogeneous set of reals of size \aleph_1* , *Fund. Math* **186** (2005), no. 1, 71–77
- [2] R. Hernández-Gutiérrez, M. Hrušák, and J. van Mill, *Countable dense homogeneity and λ -sets*, *Fund. Math* **226** (2014), no. 2, 157–172

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