

# Amenability, optimal transport and abstract ergodic theorems

Christian Rosendal<sup>1</sup>

rosendal@umd.edu

Using tools from the theory of optimal transport, four results concerning isometric actions of amenable topological groups with potentially unbounded orbits are established. Specifically, consider an amenable topological group  $G$  with no non-trivial homomorphisms to  $\mathbb{R}$ . If  $d$  is a compatible left-invariant metric on  $G$ ,  $E \subseteq G$  is a finite subset and  $\epsilon > 0$ , there is a finitely supported probability measure  $\beta$  on  $G$  so that

$$\max_{g,h \in E} W(\beta g, \beta h) < \epsilon,$$

where  $W$  denotes the *Wasserstein* or *optimal transport* distance between probability measures on the metric space  $(G, d)$ . When  $d$  is the word metric on a finitely generated group  $G$ , this strengthens a well known theorem of H. Reiter. Furthermore, when  $G$  is locally compact second countable,  $\beta$  may be replaced by an appropriate probability density  $f \in L^1(G)$ .

Also, when  $G \curvearrowright X$  is a continuous isometric action on a metric space, the space of Lipschitz functions on the quotient  $X//G$  is isometrically isomorphic to a 1-complemented subspace of the Lipschitz functions on  $X$ . And finally every continuous affine isometric action of  $G$  on a Banach space has a canonical invariant linear subspace. These results generalise previous theorems due to Schneider–Thom and Cúth–Doucha.

---

<sup>1</sup>Research partially supported by NSF award DMS 2204849.