On the center of distances

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Given a metric space X with the distance d, then the set

$$S(X) = \{\alpha : \forall_{x \in X} \exists_{y \in X} d(x, y) = \alpha\}$$

is called the *center of distances* of X. This notion occurs naturally in the following generalization of the theorem by J. von Neumann: *Suppose that sequences $\{a_n\}_{n \in \omega}$ and $\{b_n\}_{n \in \omega}$ have the same set of cluster points $C \subseteq X$, where (X, d) is a compact metric space. If $\alpha \in S(C)$, then there exists a permutation $\pi : \omega \to \omega$ such that $\lim_{n \to +\infty} d(a_n, b_{\pi(n)}) = \alpha$.* Also, it is used to study sets of subsums of some sequences of positive reals, as well to some impossibility proofs. We compute the center of distances of the Cantorval $\mathbb{X}$, which is the set of subsums of the sequence $\frac{3}{4}, \frac{1}{2}, \ldots, \frac{3}{4^n}, \frac{2}{4^n}, \ldots$, and also for some related subsets of the reals. Some of our results are: *If $q > 2$ and $a \geqslant 0$, then the center of distances of the set of subsums of a geometric sequence $\{\frac{a}{q^n}\}_{n \geqslant 1}$ consists of the terms $0, \frac{a}{q}, \frac{a}{q^2}, \ldots$; The center of distances of the Cantorval $\mathbb{X}$ consists of the terms $0, \frac{3}{4}, \frac{1}{2}, \ldots, \frac{3}{4^n}, \frac{2}{4^n}, \ldots$; The center of distances of the set $[0, \frac{5}{3}] \setminus$ Int $\mathbb{X}$ is trivial, since it consists of only Zero; Neither $[0, \frac{5}{3}] \setminus$ Int $\mathbb{X}$ nor $\mathbb{X} \setminus$ Int $\mathbb{X}$ is the set of subsums of a sequence; The center of distances of the set $\mathbb{X} \setminus$ Int $\mathbb{X}$ consists of the terms $0, \frac{1}{4}, \frac{1}{16}, \ldots, \frac{1}{4^n}, \ldots$; Let $A \subset \{\frac{2}{4^n} : n > 0\} \cup \{\frac{3}{4^n} : n > 0\} = B$ be such that $B \setminus A$ and A are infinite. Then the set of subsums of a sequence consisting of different elements of A is homeomorphic to the Cantor set.*

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