Descriptive complexity in number theory and dynamics

Bill Mance[*1], *Steve Jackson*[2]

william.mance@amu.edu.pl,
jackson@unt.edu

Informally, a real number is *normal in base b* if in its b-ary expansion, all digits and blocks of digits occur as often as one would expect them to, uniformly at random. We will denote the set of numbers normal in base b by $N(b)$. Kechris asked several questions involving descriptive complexity of sets of normal numbers. The first of these was resolved in 1994 when Ki and Linton proved that $N(b)$ is Π^0_3-complete. Further questions were resolved by Becher, Heiber, and Slaman who showed that $\bigcap_{b=2}^{\infty} N(b)$ is Π^0_3-complete and that $\bigcup_{b=2}^{\infty} N(b)$ is Σ^0_4-complete. Many of the techniques used in these proofs can be used elsewhere. We will discuss recent results where similar techniques were applied to solve a problem of Sharkovsky and Sivak and a question of Kolyada, Misiurewicz, and Snoha. Furthermore, we will discuss a recent result where the set of numbers that are continued fraction normal, but not normal in any base b, was shown to be complete at the expected level of $D_2(\Pi^0_3)$. An immediate corollary is that this set is uncountable, a result (due to Vandehey) only known previously assuming the generalized Riemann hypothesis.

[1]The first author was supported by grant 2019/34/E/ST1/00082 for the project "Set theoretic methods in dynamics and number theory," NCN (The National Science Centre of Poland).

[2]The second author was supported by NSF grant DMS-1800323.