Completeness of products

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The *completeness number* $*compl\,(\xi)$ of a convergence ξ is the least cardinal κ such that there exists a collection $\mathbb{P}$ of cardinality κ of convergence covers such that each $\mathbb{P}$-fundamental filter is ξ-adherent. A convergence is compact whenever $*compl\,(\xi) = 0$, *locally compactoid* if $*compl\,(\xi) < \aleph_0$ (equivalently, $*compl\,(\xi) = 1$), *Čech-complete* if ξ (is functionally regular) and $*compl\,(\xi) \leq \aleph_0$ (that is, ξ is *countably complete*). Extending a *Frolík theorem* (1960) to convergences, we show that if $*compl\,(\prod \Xi) < \aleph_0$, then $\prod \Xi, = \min(1, *card\,\{\xi : *card\,(\xi) > 0\})$, and, otherwise,

$$*compl\,(\textstyle\prod \Xi) = \sum_{\xi \in \Xi} *compl\,(\xi)\,.$$

In particular, $\prod \Xi$ is compact if and only if ξ is compact for each $\xi \in \Xi$ (the *Tikhonov theorem* for convergence spaces); $\prod \Xi$ is *locally compactoid* if and only if all but finitely many elements of Ξ are compact and the others are locally compactoid; $\prod \Xi$ is *countably complete* if and only if all but countably many elements of Ξ are compact and the others are countably complete.

If $\mathbb{H}$ is a class of filters then the $\mathbb{H}$-*conditional completeness number* $*\,compl_{\mathbb{H}}\,(\xi)$ is the least cardinal κ such that there exists a collection $\mathbb{P}$ of cardinality κ of convergence covers such that each $\mathbb{P}$-fundamental filter from $\mathbb{H}$ is ξ-adherent. In contrast with unconditional completeness. For example, if $\mathbb{H} = \mathbb{F}_1$, then we get a generalization of the *Oxtoby theorem* (1960) that each product of $\mathbb{F}_1$-conditionally countably complete convergences is $\mathbb{F}_1$-conditionally countably complete.